

# Achieving New Upper Bounds for the Hypergraph Duality Problem through Logic\*

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## Abstract

The hypergraph duality problem DUAL is defined as follows: given two simple hypergraphs  $\mathcal{G}$  and  $\mathcal{H}$ , decide whether  $\mathcal{H}$  consists precisely of all minimal transversals of  $\mathcal{G}$  (in which case we say that  $\mathcal{G}$  is the dual of  $\mathcal{H}$ , or, equivalently, the transversal hypergraph of  $\mathcal{H}$ ). This problem is equivalent to decide whether two given non-redundant monotone DNFs are dual. It is known that  $\overline{\text{DUAL}}$ , the complementary problem to DUAL, is in  $\text{GC}(\log^2 n, \text{PTIME})$ , where  $\text{GC}(f(n), \mathcal{C})$  denotes the complexity class of all problems that after a nondeterministic guess of  $O(f(n))$  bits can be decided (checked) within complexity class  $\mathcal{C}$ . It was conjectured that  $\overline{\text{DUAL}}$  is in  $\text{GC}(\log^2 n, \text{LOGSPACE})$ . In this paper we prove this conjecture and actually place the  $\overline{\text{DUAL}}$  problem into the complexity class  $\text{GC}(\log^2 n, \text{TC}^0)$  which is a subclass of  $\text{GC}(\log^2 n, \text{LOGSPACE})$ . We here refer to the logtime-uniform version of  $\text{TC}^0$ , which corresponds to  $\text{FO}(\text{COUNT})$ , i.e., first order logic augmented by counting quantifiers. We achieve the latter bound in two steps. First, based on existing problem decomposition methods, we develop a new nondeterministic algorithm for  $\overline{\text{DUAL}}$  that requires to guess  $O(\log^2 n)$  bits. We then proceed by a logical analysis of this algorithm, allowing us to formulate its deterministic part in  $\text{FO}(\text{COUNT})$ . From this result, by the well known inclusion  $\text{TC}^0 \subseteq \text{LOGSPACE}$ , it follows that DUAL belongs also to  $\text{DSPACE}[\log^2 n]$ . Finally, by exploiting the principles on which the proposed nondeterministic algorithm is based, we devise a deterministic algorithm that, given two hypergraphs  $\mathcal{G}$  and  $\mathcal{H}$ , computes in quadratic logspace a transversal of  $\mathcal{G}$  missing in  $\mathcal{H}$ .

## 1 Introduction

The hypergraph duality problem DUAL is one of the most mysterious and challenging decision problems of Computer Science, as its complexity has been intensively investigated for almost 40 years without any indication that the problem is tractable, nor any evidence whatsoever, why it should be intractable. Apart from a few significant upper bounds, which we review below, and a large number of restrictions that make the problem tractable, progress on pinpointing the complexity of DUAL has been rather slow. So far, the problem has been placed in relatively low complexity nondeterministic classes within coNP. It is the aim of this paper to further narrow it down by using logical methods.

**The hypergraph duality problem.** A hypergraph  $\mathcal{G}$  consists of a finite set  $V$  of vertices and a set  $E \subseteq 2^V$  of (hyper)edges.<sup>1</sup>  $\mathcal{G}$  is *simple* (or *Sperner*) if none of its edges is contained in any other of its edges. A *transversal* or *hitting set* of a hypergraph  $\mathcal{G} = \langle V, E \rangle$  is a subset of  $V$  that meets every edge in  $E$ . A transversal of  $\mathcal{G}$  is *minimal*, if none of its proper subsets is a transversal. The set of minimal transversals of a hypergraph  $\mathcal{G} = \langle V, E \rangle$  is denoted by  $\text{tr}(\mathcal{G})$ . Note that  $\text{tr}(\mathcal{G})$ , which is referred to as *the dual*<sup>2</sup> of  $\mathcal{G}$  or also as the *transversal hypergraph* of  $\mathcal{G}$ , defines itself a hypergraph on the vertex set  $V$ . The decision problem DUAL is now easily defined as follows: Given two simple hypergraphs  $\mathcal{G}$  and  $\mathcal{H}$  over vertex set  $V$ , decide whether  $\mathcal{H} = \text{tr}(\mathcal{G})$ .

An example of a hypergraph and its dual is given in Figure 1. It is well-known that the duality problem has a nice symmetry property [2]: if  $\mathcal{G}$  and  $\mathcal{H}$  are simple hypergraphs over vertex set  $V$ , then  $\mathcal{H} = \text{tr}(\mathcal{G})$  iff  $\mathcal{G} = \text{tr}(\mathcal{H})$ , and in this case  $\mathcal{G}$  and  $\mathcal{H}$  are said to be dual. The DUAL problem is also tightly related to the

\*This paper is an extended version of a paper appeared in the Proceedings of CSL-LICS 2014 [30], and also proves in a different and much more direct way part of the results of a paper appeared in the Proceedings of PODS 2013 [28].

<sup>1</sup>For future reference, in Appendix A there is a list of the definitions, notions, and notations used more frequently in this paper.

<sup>2</sup>Note that sometimes in the literature the dual hypergraph of  $\mathcal{G}$  was defined as the hypergraph derived from  $\mathcal{G}$  in which the roles of the vertices and the edges are “interchanged” (see, e.g., [2, 51]), and this is different from the transversal hypergraph. Nevertheless, lately in the literature the name “dual hypergraph” has been used with the meaning of “transversal hypergraph” (as in, e.g., [4, 17, 19, 28, 40, 41]).

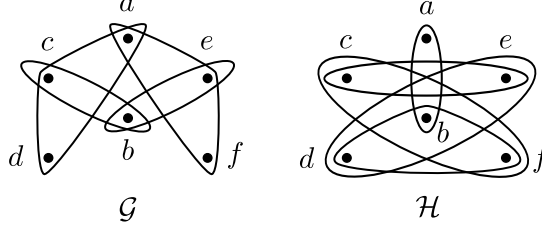


Figure 1: Hypergraph  $\mathcal{G}$  and its transversal hypergraph  $\mathcal{H}$ .

problem of actually *computing*  $tr(\mathcal{G})$  for an input hypergraph  $\mathcal{G}$ . In fact, it is known that the computation problem is feasible in total polynomial time, that is, in time polynomial in  $|\mathcal{G}| + |tr(\mathcal{G})|$ , if and only if DUAL is solvable in polynomial time [3]. These and several other properties of the duality problem are reviewed and discussed in [13, 16, 17, 33], where also many original references can be found.

**Applications of hypergraph duality.** The DUAL problem and its computational variant have a tremendous number of applications. They range from data mining [5, 6, 32, 48], functional dependency inference [29, 46, 47], and machine learning, in particular, learning monotone Boolean CNFs and DNFs with membership queries [32, 49], to model-based diagnosis [31, 50], computing a Horn approximation to a non-Horn theory [26, 39], computing minimal abductive explanations to observations [15], and computational biology, for example, discovering of metabolic networks and engineering of drugs preventing the production of toxic metabolites in cells [42, 43]. Surveys of these and other applications as well as further references can be found in [13, 14, 33, 44].

The simplest and foremost applications relevant to logic and hardware design are DNF duality testing and its computational version, DNF dualization. A pair of Boolean formulas  $f(x_1, x_2, \dots, x_n)$  and  $g(x_1, x_2, \dots, x_n)$  on propositional variables  $x_1, x_2, \dots, x_n$  are *dual* if

$$f(x_1, x_2, \dots, x_n) \equiv \neg g(\neg x_1, \neg x_2, \dots, \neg x_n).$$

A monotone DNF is *irredundant* if the set of variables in none of its disjuncts is covered by the variable set of any other disjunct. The *duality testing problem* is the problem of testing whether two irredundant monotone DNFs  $f$  and  $g$  are dual. It is well-known and easy to see that monotone DNF duality and DUAL are actually the same problem.<sup>3</sup> Two hypergraphs  $\mathcal{G}$  and  $\mathcal{H}$  are dual iff their associated DNFs  $\mathcal{G}^*$  and  $\mathcal{H}^*$  are dual, where the DNF  $\mathcal{F}^*$  associated with a hypergraph  $\mathcal{F} = \langle V, E \rangle$  is  $\bigvee_{e \in E} \bigwedge_{v \in e} v$ , where, obviously, vertices  $v \in V$  are interpreted as propositional variables. For example, the hypergraphs  $\mathcal{G}$  and  $\mathcal{H}$  of Figure 1 give rise to DNFs

$$\begin{aligned} \mathcal{G}^* &= (a \wedge c \wedge d) \vee (a \wedge e \wedge f) \vee (c \wedge b) \vee (e \wedge b), \text{ and} \\ \mathcal{H}^* &= (a \wedge b) \vee (c \wedge e) \vee (c \wedge b \wedge f) \vee (e \wedge b \wedge d) \vee (d \wedge b \wedge f), \end{aligned}$$

which are indeed mutually dual. The duality problem for irredundant monotone DNFs corresponds, in turn, to DUAL, and the problem instances  $\langle \mathcal{F}, \mathcal{G} \rangle$  and  $\langle \mathcal{F}^*, \mathcal{G}^* \rangle$  can be inter-translated by extremely low-level reductions, in particular, LOGTIME reductions, and even projection reductions. In many publications, the DUAL problem is thus right away introduced as the problem of duality checking for irredundant monotone DNFs. An equivalent problem is the problem of checking whether a monotone CNF and a monotone DNF are logically equivalent.

**Previous Complexity bounds.** DUAL is easily seen to reside in coNP. In fact, in order to show that a DUAL instance is a “no”-instance, it suffices to show that either some edge of one of the two hypergraphs is not a minimal transversal of the other hypergraph (which is feasible in polynomial time), or to find (guess and check) a missing transversal to one of the input hypergraphs. The complement  $\overline{\text{DUAL}}$  of DUAL is therefore in NP. In their landmark paper, Fredman and Khachiyan [22] have shown that DUAL is in  $\text{DTIME}[n^{o(\log n)}]$ , more precisely, that it is contained in  $\text{DTIME}[n^{4\chi(n)+O(1)}]$ , where  $\chi(n)$  is defined by  $\chi(n)^{\chi(n)} = n$ . Note that  $\chi(n) \sim \log n / \log \log n = o(\log n)$ .

Let  $\text{GC}(f(n), \mathcal{C})$  denote the complexity class of all problems that after a nondeterministic guess of  $O(f(n))$  bits can be decided (checked) in complexity class  $\mathcal{C}$ . Eiter, Gottlob, and Makino [16], and independently, Kavvadias and Stavropoulos [38] have shown that  $\overline{\text{DUAL}}$  is in  $\text{GC}(\log^2 n, \text{PTIME})$ ; note that this class is also known as  $\beta_2\text{P}$ , see [27].

Recently, in a precursor of the present paper [28], the nondeterministic bound for  $\overline{\text{DUAL}}$  was further pushed down to  $\text{GC}(\log^2 n, \llbracket \text{LOGSPACE}_{\text{pol}} \rrbracket^{\log})$ , see Figure 2a.

<sup>3</sup>In fact, in the literature, the hypergraph transversal problem was tackled interchangeably from the perspective of monotone Boolean formula dualization, or from the perspective of hypergraphs. Readers wanting to know more about the relationships between some of the different perspectives adopted in the literature to deal with the DUAL problem are referred to Appendix B.

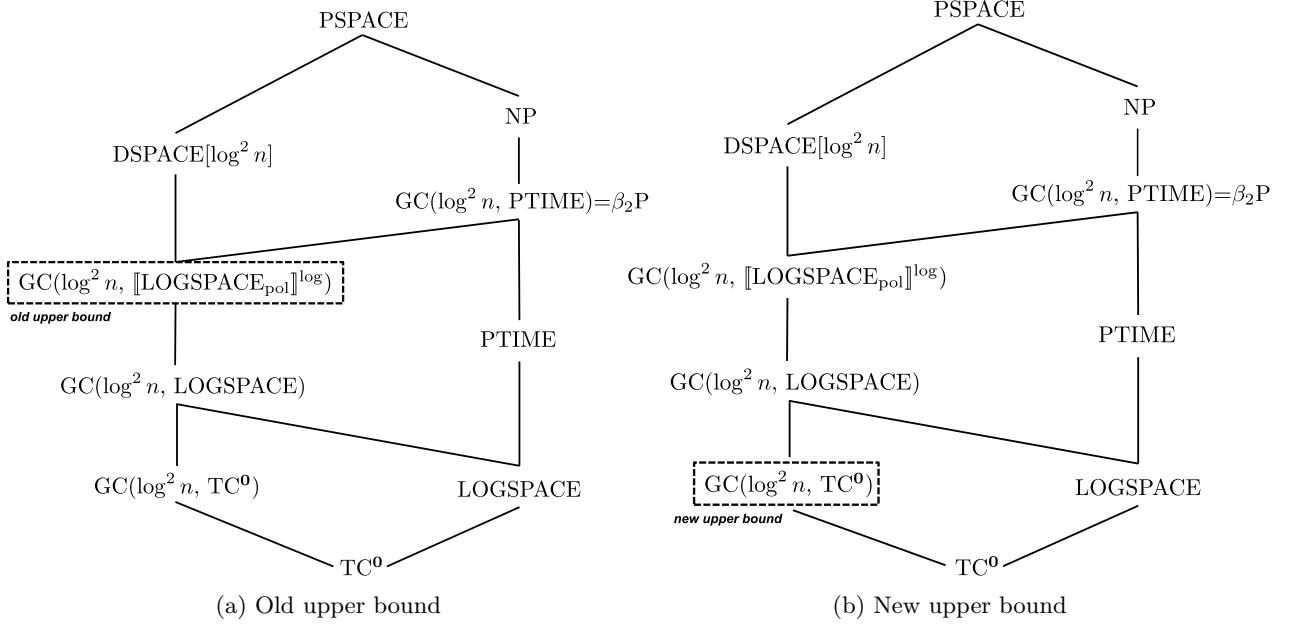


Figure 2: Complexity bound improvement obtained in this paper.

A precise definition of  $\llbracket \text{LOGSPACE}_{\text{pol}} \rrbracket^{\log}$  is given in [28]. We will not make use of this class in the technical part of the present paper. Informally,  $\llbracket \text{LOGSPACE}_{\text{pol}} \rrbracket^{\log}$  contains those problems  $\pi$  for which there exists a logspace-transducer  $T$ , a polynomial  $p$ , and a function  $f$  in  $O(\log n)$ , such that each  $\pi$ -instance  $I$  of size  $n = |I|$  can be reduced by the  $f(n)$ -fold composition  $T^{f(n)}$  of  $T$  to a decision problem in LOGSPACE, where the size of all intermediate results  $T^i(I)$ , for  $1 \leq i \leq f(n)$  is polynomially bounded by  $p(n)$ . For the relationship of  $\text{GC}(\log^2 n, \llbracket \text{LOGSPACE}_{\text{pol}} \rrbracket^{\log})$  to other classes, see Figure 2a. In particular, in [28], it was shown that  $\text{GC}(\log^2 n, \llbracket \text{LOGSPACE}_{\text{pol}} \rrbracket^{\log})$  is not only a subclass of  $\text{GC}(\log^2 n, \text{PTIME})$ , but also of  $\text{DSPACE}[\log^2 n]$ , i.e., of quadratic logspace. Therefore, as also proven more directly in the present paper (Corollary 4.10), DUAL is in  $\text{DSPACE}[\log^2 n]$ , and it is thus most unlikely for DUAL to be PTIME-hard, which answered a previously long standing question. Given that PTIME and  $\text{DSPACE}[\log^2 n]$  are believed to be incomparable, it is also rather unlikely that DUAL is closely related to another interesting logical problem of open complexity, namely, to validity-checking for the modal  $\mu$ -calculus, or, equivalently, to the winner determination problem for parity games [34, 36], as these latter problems are PTIME-hard, but in  $\text{NP} \cap \text{coNP}$ .

**Main complexity problem tackled.** In [28] it was asked whether the upper bound of  $\text{GC}(\log^2 n, \llbracket \text{LOGSPACE}_{\text{pol}} \rrbracket^{\log})$  could be pushed further downwards, and the following conjecture was made:

*Conjecture ([28])*  $\overline{\text{DUAL}} \in \text{GC}(\log^2 n, \text{LOGSPACE})$ .

It was unclear, however, how to prove this conjecture based on the algorithms and methods used in [28]. There, a problem decomposition strategy by Boros and Makino [4] was used, that decomposed an original DUAL instance into a conjunction of smaller instances according to a specific conjunctive self-reduction. Roughly, this strategy constructs a decomposition tree of logarithmic depth for DUAL, each of whose nodes represents a sub-instance of the original instance; more details on decomposition trees are given in Section 3. To prove that the original instance is a “no”-instance (and thus a “yes”-instance of  $\overline{\text{DUAL}}$ ), it is sufficient to guess, in that tree, a path  $\Pi$  from the root to a single node  $v$  associated with a “no”-sub-instance that can be recognized as such in logarithmic space. Guessing the path to  $v$  can be easily done using  $O(\log^2 n)$  nondeterministic bits, but it is totally unclear how to actually *compute* the sub-instance associated with node  $v$  in LOGSPACE. In fact, it seems that the only way to compute the sub-instance at node  $v$  is to compute—at least implicitly—all intermediate DUAL instances arising on the path from the root to the decomposition node  $v$ . This seems to require a logarithmic composition of LOGSPACE transducers, and thus a computation in the complexity class  $\llbracket \text{LOGSPACE}_{\text{pol}} \rrbracket^{\log}$ . It was therefore totally unclear how  $\llbracket \text{LOGSPACE}_{\text{pol}} \rrbracket^{\log}$  could be replaced by its subclass LOGSPACE, and new methods were necessary to achieve this goal.

**New results: Logic to the rescue.** To attack the problem, we studied various alternative decomposition strategies for DUAL, among them the strategy of Gaur [24], which also influenced the method of Boros and Makino [4]. In the present paper, we build on Gaur’s original strategy, as it appears to be the best starting point for our purposes. However, Gaur’s method still does not directly lead to a guess-and-check algorithm whose checking procedure is in LOGSPACE, and thus new techniques needed to be developed.

In a first step, by building creatively on Gaur’s deterministic decomposition strategy [24], we develop a new nondeterministic guess-and-check algorithm ND-NOTDUAL for  $\overline{\text{DUAL}}$ , that is specifically geared towards a computationally simple checking part. In particular, the checking part of ND-NOTDUAL avoids certain obstructive steps that would require more memory than just plain LOGSPACE, such as the successive minimization of hypergraphs in sub-instances of the decomposition (as used by Boros and Makino [4]) and the performance of counting operations between subsequent decomposition steps so to determine sets of vertices to be included in a new transversal (as used by Gaur [24]). Our new algorithm is thus influenced by Gaur’s, but differs noticeably from it, as well as from the algorithm of Boros and Makino.

In a second step, we proceed with a careful logical analysis of the checking part of ND-NOTDUAL. We transform all sub-tasks of ND-NOTDUAL into logical formulas. However, it turns out that first order logic (FO) is not sufficient, as an essential step of the checking phase of ND-NOTDUAL is to check for specific hypergraph vertices  $v$  whether  $v$  is contained in at least half of the hyperedges of some hypergraph. To account for this, we need to resort to FO(COUNT), which augments FO with counting quantifiers. Note that we could have used in a similar way FOM, i.e., FO augmented by majority quantifiers, as FO(COUNT) and FOM have the same expressive power [35]. By putting all pieces together, we succeed to describe the entire checking phase by a single fixed FO(COUNT) formula that has to be evaluated over the input  $\overline{\text{DUAL}}$  instance. Note that FO(COUNT) model-checking is complete for logtime-uniform  $\text{TC}^0$ .

In summary, by putting the guessing and checking parts together, we achieve as main theorem a complexity result that is actually better than the one conjectured:

*Theorem.*  $\overline{\text{DUAL}} \in \text{GC}(\log^2 n, \text{TC}^0)$ .

By the well-known inclusion  $\text{TC}^0 \subseteq \text{LOGSPACE}$ , we immediately obtain a corollary that proves the above mentioned conjecture:

*Corollary A.*  $\overline{\text{DUAL}} \in \text{GC}(\log^2 n, \text{LOGSPACE})$ .

Moreover, by the inclusion  $\text{GC}(\log^2 n, \text{LOGSPACE}) \subseteq \text{DSPACE}[\log^2 n]$ , and the fact that  $\text{DSPACE}[\log^2 n]$  is closed under complement, we can easily obtain as a simple corollary that:

*Corollary B.*  $\text{DUAL} \in \text{DSPACE}[\log^2 n]$ .

To conclude, by an easy adaptation of our algorithm ND-NOTDUAL we devise a simple deterministic algorithm COMPUTENT to *compute* a new (not necessarily minimal) transversal in quadratic logspace.

**Significance of the new results and directions for future research.** The progress achieved in this paper is summarized in Figure 2, whose left part (Figure 2a) shows the previous state of knowledge about the complexity, while the right part (Figure 2b) depicts the current state of knowledge we have achieved. We have significantly narrowed down the “search space” for the precise complexity of  $\text{DUAL}$  (or  $\overline{\text{DUAL}}$ ). We believe that our new results are of value to anybody studying the complexity of this interesting problem. In particular, the connection to logic opens new avenues for such studies. First, our results show where to dig for tighter bounds. It may be rewarding to study subclasses of  $\text{GC}(\log^2 n, \text{TC}^0)$ , and in particular, *logically defined subclasses* that replace  $\text{TC}^0$  by low-level prefix classes of FO(COUNT). Classes of this type can be found in [8, 9, 21, 52, 53]. Second, the membership of  $\overline{\text{DUAL}}$  in  $\text{GC}(\log^2 n, \text{TC}^0)$  provides valuable information for those trying to prove hardness results for  $\text{DUAL}$ , i.e., to reduce some presumably intractable problem  $X$  to  $\text{DUAL}$ . Our results restrict the search space to be explored to hunt for such a problem  $X$ . Moreover, given that LOGSPACE is not known to be in  $\text{GC}(\log^2 n, \text{TC}^0)$ , and given that it is not generally believed that  $\text{GC}(\log^2 n, \text{TC}^0)$  contains LOGSPACE-hard problems (under logtime reductions), our results suggest that LOGSPACE-hard problems are rather unlikely to reduce to  $\text{DUAL}$ , and that it may thus be advisable to look for a problem  $X$  that is not (known to be) LOGSPACE-hard, in order to find a lower bound for  $\text{DUAL}$ . Our new results are of theoretical nature. This does not rule out the possibility that they may be used for improving practical algorithms, but this has yet to be investigated. Finally, we believe that the methods presented in this paper are a compelling example of how logic and descriptive complexity theory can be used together with suitable problem decomposition methods to achieve new complexity results for a concrete decision problem.

**Organization of the paper.** After some preliminaries in Section 2, we discuss problem decomposition strategies and introduce the concept of a decomposition tree for  $\overline{\text{DUAL}}$  in Section 3. Based on this, in Section 4 we present the ND-NOTDUAL nondeterministic algorithm for  $\overline{\text{DUAL}}$ , prove it correct, and then analyze this algorithm to derive our main complexity results. To conclude, by exploiting the method used in the nondeterministic algorithm, we present a deterministic algorithm to actually compute a new (not necessarily minimal) transversal in quadratic logspace.

## 2 Preliminaries

In what follows, we will often identify a hypergraph  $\mathcal{G} = \langle V, E \rangle$  with its edge-set and vice-versa. By writing  $G \in \mathcal{G}$  we mean  $G \in E$ , and by writing  $\mathcal{G} = E$  we mean that  $\mathcal{G} = \langle \bigcup_{G \in E} G, E \rangle$ . Moreover, if not stated otherwise, hypergraphs have the same set of vertices, which is denoted by  $V$ , and each vertex of a hypergraph belongs to at least one of its edges. If  $\mathcal{G} = \langle V, E \rangle$  and  $\mathcal{H} = \langle W, F \rangle$  are two hypergraphs, we denote by  $\mathcal{G} \subseteq \mathcal{H}$  the fact that  $V \subseteq W$  and  $E \subseteq F$ .  $|\mathcal{G}|$  denotes the number of edges of  $\mathcal{G}$ , and, given two hypergraphs  $\mathcal{G}$  and  $\mathcal{H}$ ,  $m$  is the total number  $|\mathcal{G}| + |\mathcal{H}|$  of edges of  $\mathcal{G}$  and  $\mathcal{H}$ . By  $\|\mathcal{G}\|$  we denote the size of the hypergraph  $\mathcal{G}$ , that is the space (in terms of the number of bits) required to represent  $\mathcal{G}$ . It is reasonable to assume that a hypergraph  $\mathcal{G}$  is represented through the adjacency lists of its edges (i.e., each edge  $G$  of  $\mathcal{G}$  is represented through the list of the vertices belonging to  $G$ ). It is easy to see that  $|V| \leq \|\mathcal{G}\|$ , and  $|\mathcal{G}| \leq \|\mathcal{G}\|$ . We denote by  $N = \|\mathcal{G}\| + \|\mathcal{H}\|$  the size of the input of the DUAL problem.

We say that  $\mathcal{G}$  is an *empty hypergraph* if  $\mathcal{G} = \emptyset$ , and  $\mathcal{G}$  is an *empty-edge hypergraph* if  $\mathcal{G} = \{\emptyset\}$ . In this paper, unless it is explicitly stated, we assume that hypergraphs are neither empty nor contain the empty-edge. Observe that, if  $\mathcal{G} = \emptyset$ , then any set of vertices is a transversal of  $\mathcal{G}$ . For this reason, there is only one minimal transversal of  $\mathcal{G}$  and it is the empty set, because any strict superset of the empty set, although it is a transversal of  $\mathcal{G}$ , it is not minimal. On the other hand, if  $\mathcal{G} = \{\emptyset\}$ , then there is no transversal of  $\mathcal{G}$  at all, because any set of vertices, even the empty one, has an empty intersection with the empty-edge of  $\mathcal{G}$ . Hence, by definition, the dual of the empty hypergraph is the empty-edge hypergraph, and vice-versa [13]. Two hypergraphs  $\mathcal{G}$  and  $\mathcal{H}$  are *trivially dual*, if one of them is the empty hypergraph and the other is the empty-edge one.

Given a hypergraph  $\mathcal{G}$  and a set of vertices  $T$ , a vertex  $v$  is *critical* in  $T$  (w.r.t.  $\mathcal{G}$ ) if there exists an edge  $G \in \mathcal{G}$  such that  $G \cap T = \{v\}$ . We say that  $G$  witnesses the criticality of  $v$  in  $T$ . Observe that, if  $v$  is a critical vertex in  $T$ ,  $v$  may have many different witnesses of its criticality, that is, more than one edge of  $\mathcal{G}$  can intersect  $T$  only on  $v$ .

A set of vertices  $S$  is an *independent set* of a hypergraph  $\mathcal{G}$  if, for all  $G \in \mathcal{G}$ ,  $G \not\subseteq S$ . If  $\mathcal{G}$  and  $\mathcal{H}$  are two hypergraphs, a set of vertices  $T$  is a *new transversal* of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$  if  $T$  is a transversal of  $\mathcal{G}$ , and  $T$  is also an independent set of  $\mathcal{H}$ . Intuitively, a new transversal  $T$  of  $\mathcal{G}$  is a transversal of  $\mathcal{G}$  for which if  $T' \subseteq T$  is a transversal of  $\mathcal{G}$ , then  $T' \notin \mathcal{H}$ . Observe that a new transversal does not necessarily need to be a minimal transversal.

Given a hypergraph  $\mathcal{G}$  and a set  $S$  of vertices, as in [4, 19], we define hypergraphs  $\mathcal{G}_S = \langle S, \{G \in \mathcal{G} \mid G \subseteq S\} \rangle$ , and  $\mathcal{G}^S = \langle S, \min(\{G \cap S \mid G \in \mathcal{G}\}) \rangle$ , where  $\min(\mathcal{H})$ , for any hypergraph  $\mathcal{H}$ , denotes the set of inclusion minimal edges of  $\mathcal{H}$ . Observe that  $\mathcal{G}^S$  is always a simple hypergraph, and that if  $\mathcal{G}$  is simple, then so is  $\mathcal{G}_S$ . If  $\emptyset \in \mathcal{G}$  (i.e., hypergraph  $\mathcal{G}$  contains the empty-edge), then  $\min(\mathcal{G}) = \{\emptyset\}$ .

In the following we state the main properties about transversals to be used in this paper. Some of the following properties are already known (see, e.g., [2, 4, 13, 19, 22, 23, 25]), and some of them were stated over Boolean formulas. We state over hypergraphs all the properties relevant for us, and for completeness we prove them. The proofs are reported in Appendix C.

**Lemma 2.1.** *Let  $\mathcal{H}$  be a hypergraph, and let  $T \subseteq V$  be a transversal of  $\mathcal{H}$ . Then,  $T$  is a minimal transversal of  $\mathcal{H}$  if and only if every vertex  $v \in T$  is critical (and hence there exists an edge  $H_v \in \mathcal{H}$  witnessing so).*

If the vertex set is  $V$  and  $T \subseteq V$ , let  $\bar{T}$  denote  $V \setminus T$ , i.e., the *complement* of  $T$  in  $V$ .

**Lemma 2.2.** *Let  $\mathcal{G}$  and  $\mathcal{H}$  be two hypergraphs. A set of vertices  $T \subseteq V$  is a new transversal of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$  if and only if  $\bar{T}$  is a new transversal of  $\mathcal{H}$  w.r.t.  $\mathcal{G}$ .*

We say that hypergraphs  $\mathcal{G}$  and  $\mathcal{H}$  satisfy the *intersection property* if all the edges of  $\mathcal{G}$  are transversals of  $\mathcal{H}$ , and thus, vice-versa all the edges of  $\mathcal{H}$  are transversal of  $\mathcal{G}$ . Note here that, for the intersection property to hold, the edges of one hypergraph are *not* required to be *minimal* transversal of the other.

**Lemma 2.3.** *Let  $\mathcal{G}$  and  $\mathcal{H}$  be two hypergraphs. Then,  $\mathcal{G}$  and  $\mathcal{H}$  are dual if and only if  $\mathcal{G}$  and  $\mathcal{H}$  are simple, satisfy the intersection property, and there is no new transversal of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$ .*

## 3 Decomposing the DUAL problem

### 3.1 Decomposition principles

An approach to recognize “no”-instances  $\langle \mathcal{G}, \mathcal{H} \rangle$  of DUAL is to find a new transversal of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$ , i.e., a transversal of  $\mathcal{G}$  that is also an independent set of  $\mathcal{H}$ . In fact, many algorithms in the literature follow this approach (see, e.g., [4, 16, 19, 22, 24, 38]). These algorithms try to build such a new transversal by successively

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<sup>4</sup>We will often omit “w.r.t.  $\mathcal{H}$ ” when the hypergraph  $\mathcal{H}$  we are referring to is understood.

including vertices in and excluding vertices from a candidate for a new transversal. To give an example, the classical algorithm “A” of Fredman and Khachiyan [22] tries to include a vertex  $v$  in a candidate for a new transversal, and if this does not result in a new transversal, then  $v$  is excluded. Moreover if the exclusion of  $v$  does not lead to a new transversal, then no new transversal exists which is coherent with the choices having been made before considering the vertex  $v$ . (If  $v$  is the first vertex that has been considered, then there is no new transversal at all.)

We speak about *included* and *excluded* vertices because most of the algorithms proposed in the literature implicitly or explicitly keep track of two sets: the set of the vertices considered included in and the set of the vertices considered excluded from the attempted new transversal. Similarly, those algorithms working on Boolean formulas keep track of the truth assignment: the variables to which **true** has been assigned, those to which **false** has been assigned, and (obviously) those to which no Boolean value has been assigned yet.

If  $\mathcal{G}$  and  $\mathcal{H}$  are two hypergraphs, an *assignment*  $\sigma = \langle In, Ex \rangle$  is a pair of subsets of  $V$  such that  $In \cap Ex = \emptyset$ . Intuitively, the set  $In$  contains the vertices considered included in (or, inside) an attempted new transversal  $T \supseteq In$  of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$ , while the set  $Ex$  contains the vertices considered excluded from (or, outside)  $T$  (i.e.,  $T \cap Ex = \emptyset$ ). We say that a vertex  $v \in V$  is free in an assignment  $\sigma = \langle In, Ex \rangle$ , if  $v \notin In$  and  $v \notin Ex$ . Note that the *empty assignment*  $\sigma_\varepsilon = \langle \emptyset, \emptyset \rangle$  is a valid assignment. Given assignments  $\sigma_1 = \langle In_1, Ex_1 \rangle$  and  $\sigma_2 = \langle In_2, Ex_2 \rangle$ , if  $In_1 \cap Ex_2 = \emptyset$  and  $Ex_1 \cap In_2 = \emptyset$ , we denote by  $\sigma_1 + \sigma_2 = \langle In_1 \cup In_2, Ex_1 \cup Ex_2 \rangle$  the extension of  $\sigma_1$  with  $\sigma_2$ . An assignment  $\sigma_2 = \langle In_2, Ex_2 \rangle$  is said to be an *extension* of the assignment  $\sigma_1 = \langle In_1, Ex_1 \rangle$ , denoted by  $\sigma_1 \sqsubseteq \sigma_2$ , whenever  $In_1 \subseteq In_2$  and  $Ex_1 \subseteq Ex_2$ . If  $\sigma_1 \sqsubseteq \sigma_2$  and  $In_1 \subset In_2$  or  $Ex_1 \subset Ex_2$  we say that  $\sigma_2$  is a *proper extension* of  $\sigma_1$ , denoted by  $\sigma_1 \subset \sigma_2$ .

Given a set of vertices  $S \subseteq V$ , the associated assignment is  $\sigma_S = \langle S, \overline{S} \rangle$ . We say that an assignment  $\sigma = \langle In, Ex \rangle$  is *coherent* with a set of vertices  $S$ , and vice-versa, whenever  $\sigma \sqsubseteq \sigma_S$ . This is tantamount to  $In \subseteq S$  and  $Ex \subseteq \overline{S}$  (or, equivalently,  $Ex \cap S = \emptyset$ ). With a slight abuse of notation we denote that an assignment  $\sigma$  is coherent with a set  $S$  by  $\sigma \sqsubseteq S$ . Observe that, by Lemma 2.2, if  $\sigma = \langle In, Ex \rangle$  is coherent with a new transversal  $T$  of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$ , then the *reversed assignment*  $\overline{\sigma} = \langle Ex, In \rangle$  is coherent with the new transversal  $\overline{T}$  of  $\mathcal{H}$  w.r.t.  $\mathcal{G}$ . Intuitively, this means that given an assignment  $\sigma = \langle In, Ex \rangle$ , the set  $In$  is (a subset of) an attempted new transversal of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$ , and, symmetrically, the set  $Ex$  is (a subset of) an attempted new transversal of  $\mathcal{H}$  w.r.t.  $\mathcal{G}$ .

Most algorithms proposed in the literature essentially successively try different assignments by successively extending in different ways the current assignment. Each extension performed induces a “reduced” instance of DUAL on which the algorithm is recursively invoked. Intuitively, the size reduction of the instance happens for two reasons. Including vertices in the new transversal of  $\mathcal{G}$  increases the number of edges of  $\mathcal{G}$  met by the new transversal under construction, and hence there is no need to consider these edges any longer. Symmetrically, excluding vertices from the new transversal of  $\mathcal{G}$  increases the number of edges of  $\mathcal{H}$  certainly not contained in the new transversal under construction, and hence, again, there is no need to consider these edges any longer.

Let  $\mathcal{I} = \langle \mathcal{G}, \mathcal{H} \rangle$  be an instance of DUAL. While constructing a new transversal of  $\mathcal{G}$ , when the assignment  $\sigma = \langle In, Ex \rangle$  is considered let us denote by  $\mathcal{I}_\sigma = \langle \mathcal{G}(\sigma), \mathcal{H}(\sigma) \rangle = \langle (\mathcal{G}_{V \setminus In})^{V \setminus (In \cup Ex)}, (\mathcal{H}_{V \setminus Ex})^{V \setminus (In \cup Ex)} \rangle$  the reduced instance derived from  $\mathcal{I}$  and induced by  $\sigma$ . Note that both  $\mathcal{G}(\sigma)$  and  $\mathcal{H}(\sigma)$  are simple by definition because they undergo a minimization operation. Intuitively, since we are interested in finding new transversals of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$ , we can avoid to analyze and further extend an assignment  $\sigma = \langle In, Ex \rangle$  for which  $In$  is not an independent set of  $\mathcal{H}$  or  $Ex$  is not an independent set of  $\mathcal{G}$ . In fact, on the one hand, if  $Ex$  is not an independent set of  $\mathcal{G}$ , then no set of vertices  $T$  coherent with  $\sigma$  can be a transversal of  $\mathcal{G}$  (because, by  $\sigma \sqsubseteq T$ ,  $T$  and  $Ex$  are disjoint). On the other hand, if  $In$  is not an independent set of  $\mathcal{H}$ , then no set of vertices  $T$  coherent with  $\sigma$  can be an independent set of  $\mathcal{H}$ , and hence a new transversal of  $\mathcal{G}$  (because, by  $\sigma \sqsubseteq T$ ,  $In \subseteq T$ ). To this purpose, for an assignment  $\sigma = \langle In, Ex \rangle$ , if there exists an edge  $H \in \mathcal{H}$  with  $H \subseteq In$  or an edge  $G \in \mathcal{G}$  with  $G \subseteq Ex$ , we say that  $In$  and  $Ex$  are *covering*, respectively. We also say that  $\sigma = \langle In, Ex \rangle$  is a *covering assignment* if  $In$  or  $Ex$  are covering. For future reference, let us highlight the just mentioned property in the following lemma.

**Lemma 3.1.** *Let  $\mathcal{G}$  and  $\mathcal{H}$  be two hypergraphs, and let  $\sigma = \langle In, Ex \rangle$  be an assignment.*

- (a) *If  $T$  is a transversal of  $\mathcal{G}$  coherent with  $\sigma$ , then  $Ex$  is not covering;*
- (b) *If  $T$  is an independent set of  $\mathcal{H}$  coherent with  $\sigma$ , then  $In$  is not covering.*

*Hence, if  $T$  is a new transversal of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$  coherent with  $\sigma$ , then  $\sigma$  is not covering.*

The advantage of decomposing an original instance of DUAL in multiple sub-instances is that of generating smaller sub-instances for which it is computationally easier to check their duality. In fact, many algorithms proposed in the literature decompose the original instance into smaller sub-instances for which the duality test is feasible in PTIME or even in subclasses of it. The following property of DUAL sub-instances is of key importance for the correctness of all the approaches tackling DUAL through decomposition techniques.

**Lemma 3.2.** *Two hypergraphs  $\mathcal{G}$  and  $\mathcal{H}$  are dual if and only if  $\mathcal{G}$  and  $\mathcal{H}$  are simple, satisfy the intersection property, and, for all assignments  $\sigma$ ,  $\mathcal{G}(\sigma)$  and  $\mathcal{H}(\sigma)$  are dual (or, equivalently, there is no new transversal of  $\mathcal{G}(\sigma)$  w.r.t.  $\mathcal{H}(\sigma)$ ).*

Lemma 3.2 can be easily proven by exploiting Lemma 2.3 and the equivalence between the hypergraph transversal problem and the duality problem of non-redundant monotone Boolean CNF/DNF formulas, where (partial) assignments here considered correspond to partial Boolean truth assignments. However, in Appendix D, we provide a detailed proof of the previous lemma without resorting to the equivalence with Boolean formulas.

Even though checking the duality of sub-instances of an initial instance of DUAL can be computationally easier, it is evident that, in order to find a new transversal of  $\mathcal{G}$ , naively trying all the possible non-covering assignments would require exponential time. Nevertheless, as already mentioned, there are deterministic algorithms solving DUAL in sub-exponential time. To meet that time bound, those algorithms do not try all the possible combinations of assignments, but they consider specific assignment extensions.

Common approaches—referred to as extension-types—to extend a currently considered assignment  $\sigma = \langle In, Ex \rangle$  to an assignment  $\sigma'$  are:

- (i) include a vertex  $v$  in the new transversal  $T$  of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$ , i.e.,  $\sigma' = \sigma + \langle \{v\}, \emptyset \rangle$ ;
- (ii) include a vertex  $v$  as a critical vertex in the new transversal  $T$  of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$  with edge  $G \in \mathcal{G}(\sigma)$  witnessing  $v$ 's criticality, i.e.,  $\sigma' = \sigma + \langle \{v\}, G \setminus \{v\} \rangle$  (observe that if  $G \in \mathcal{G}(\sigma)$ , then  $G \cap In = \emptyset$ );
- (iii) exclude a vertex  $v$  from the new transversal  $T$  of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$  (or, equivalently, include a vertex  $v$  in the new transversal  $T$  of  $\mathcal{H}$  w.r.t.  $\mathcal{G}$ ), i.e.,  $\sigma' = \sigma + \langle \emptyset, \{v\} \rangle$ ;
- (iv) include a vertex  $v$  as a critical vertex in the new transversal  $T$  of  $\mathcal{H}$  w.r.t.  $\mathcal{G}$  with edge  $H \in \mathcal{H}(\sigma)$  witnessing  $v$ 's criticality, i.e.,  $\sigma' = \sigma + \langle H \setminus \{v\}, \{v\} \rangle$  (observe that if  $H \in \mathcal{H}(\sigma)$ , then  $H \cap Ex = \emptyset$ ).

The size reduction attained in the considered sub-instances tightly depends on the assignment extension performed, and in particular on the frequencies with which vertices belong to the edges of  $\mathcal{G}(\sigma)$  and  $\mathcal{H}(\sigma)$ . We will analyze this in details below.

### 3.2 Assignment trees and the definition of $\mathcal{T}(\mathcal{G}, \mathcal{H})$

Most algorithms proposed in the literature adopt their own specific assignment extensions in specific sequences. The assignments successively considered during the recursive execution of the algorithms can be analyzed through a tree-like structure. Intuitively, each node of the tree can be associated with a tried assignment, and nodes of the tree are connected when their assignments are one the direct extension of the other. We can call these trees *assignment trees*.

Inspired by the algorithm proposed by Gaur [24],<sup>5</sup> we will now describe the construction of a general assignment tree  $\mathcal{T}(\mathcal{G}, \mathcal{H})$  that simultaneously represents all possible decompositions of an input DUAL instance  $\langle \mathcal{G}, \mathcal{H} \rangle$  according to (assignment extensions directly derived from) the extension-types (ii) and (iii). This tree is of super-polynomial size. However, it will be shown later that whenever  $\mathcal{H} \neq tr(\mathcal{G})$ , then there must exist in this tree a path of length  $O(\log N)$  whose end-node can be recognized with low computational effort as a node witnessing the non-duality of hypergraphs  $\mathcal{G}$  and  $\mathcal{H}$ .

Intuitively, each node  $p$  of the tree  $\mathcal{T}(\mathcal{G}, \mathcal{H})$  is associated with an assignment  $\sigma_p$ . In particular, the root is labeled with the empty assignment. Node  $p$  of the tree has a child  $q$  for each assignment  $\sigma_q$  that can be obtained from  $\sigma_p$  through an elementary extension of type (ii) or (iii). The edge  $(p, q)$  is then labeled by precisely this extension.

For our purposes, drawing upon the algorithm of Gaur, for each node  $p$  of the tree whose assignment is  $\sigma_p = \langle In_p, Ex_p \rangle$  we do not (explicitly) consider the sub-instance  $\mathcal{I}_p = \langle \mathcal{G}(\sigma_p), \mathcal{H}(\sigma_p) \rangle$ . We refer instead to the following sets:

- $Sep_{\mathcal{G}, \mathcal{H}}(\sigma_p) = \{G \in \mathcal{G} \mid G \cap In_p = \emptyset\}$ , the set of all edges of  $\mathcal{G}$  *not met* by (or, equivalently, *separated* from)  $\sigma_p$ ; and
- $Com_{\mathcal{G}, \mathcal{H}}(\sigma_p) = \{H \in \mathcal{H} \mid H \cap Ex_p = \emptyset\}$ , the set of all edges of  $\mathcal{H}$  *compatible* with  $\sigma_p$ .

<sup>5</sup>Readers can find in Appendix E a *deterministic* algorithm, based on that of Gaur [24] (see also [25]), deciding hypergraph duality. Note that the original algorithm proposed by Gaur aims instead at deciding *self*-duality of DNF Boolean formulas. It is from the algorithm reported in the appendix that we have taken ideas to devise our nondeterministic algorithm. Note that the exposition in Appendix E builds up on concepts, definitions, and lemmas discussed in Sections 3.2 and 3.3.

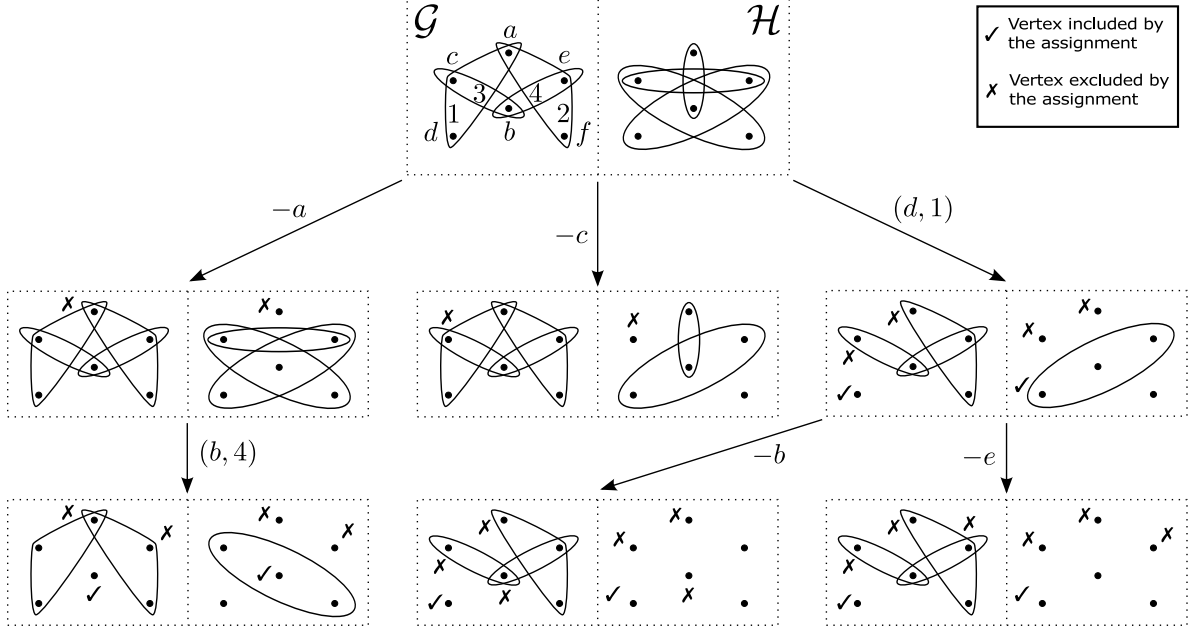


Figure 3: A subtree of the decomposition tree  $\mathcal{T}(\mathcal{G}, \mathcal{H})$ .

We will often omit the subscript “ $\mathcal{G}, \mathcal{H}$ ” of  $Sep_{\mathcal{G}, \mathcal{H}}(\sigma_p)$  and  $Com_{\mathcal{G}, \mathcal{H}}(\sigma_p)$  when the hypergraphs  $\mathcal{G}$  and  $\mathcal{H}$  we are referring to are understood. The sets  $Sep(\sigma_p)$  and  $Com(\sigma_p)$  are very similar to  $\mathcal{G}(\sigma_p)$  and  $\mathcal{H}(\sigma_p)$ , respectively. However, roughly speaking, unlike  $\mathcal{G}(\sigma_p)$  and  $\mathcal{H}(\sigma_p)$ , the edges in  $Sep(\sigma_p)$  and  $Com(\sigma_p)$  are neither “projected” over the free vertices of  $\sigma_p$ , nor minimized to obtain simple hypergraphs. In fact,  $Sep(\sigma_p)$  and  $Com(\sigma_p)$  are (more or less) equivalent to  $\mathcal{G}_{V \setminus In_p}$  and to  $\mathcal{H}_{V \setminus Ex_p}$ , respectively (and not to  $(\mathcal{G}_{V \setminus In_p})^{V \setminus (In_p \cup Ex_p)}$  and to  $(\mathcal{H}_{V \setminus Ex_p})^{V \setminus (In_p \cup Ex_p)}$ , respectively). The difference lies in the fact that  $Sep(\sigma_p)$  and  $Com(\sigma_p)$  are formally defined as sets of sets of vertices, while  $\mathcal{G}_{V \setminus In_p}$  and  $\mathcal{H}_{V \setminus Ex_p}$  are hypergraphs.

More formally, let  $\mathcal{T}(\mathcal{G}, \mathcal{H}) = \langle N, A, r, \sigma, \ell \rangle$  be a tree whose nodes  $N$  are labeled by a function  $\sigma$ , and whose edges  $A$  are labeled by a function  $\ell$ . The root  $r \in N$  of the tree is labeled with the empty assignment  $\sigma_\varepsilon = \langle \emptyset, \emptyset \rangle$ . Each node  $p$  is labeled with the assignment  $\sigma_p = \langle In_p, Ex_p \rangle$  (specified below). The leaves of  $\mathcal{T}(\mathcal{G}, \mathcal{H})$  are all the nodes  $p$  whose assignment  $\sigma_p$  is covering or has no free vertex (remember that, by the converse of Lemma 3.1, there is no benefit in considering (and further extending) covering assignments). Each non-leaf node  $p$  of  $\mathcal{T}(\mathcal{G}, \mathcal{H})$  has precisely the following children:

- For each free vertex  $v$  of  $\sigma_p$ ,  $p$  has a child  $q$  such that  $\sigma_q = \sigma_p + \langle \emptyset, \{v\} \rangle$ , and such that the edge connecting  $p$  to  $q$  is labeled  $-v$ .
- For each free vertex  $v$  of  $\sigma_p$ , and for each  $G \in Sep(\sigma_p)$  where  $v \in G$ ,  $p$  has a child  $q$  such that  $\sigma_q = \sigma_p + \langle \{v\}, G \setminus \{v\} \rangle$ , and such that the edge connecting  $p$  to  $q$  is labeled  $(v, G)$ .

Observe that, by this definition of  $\mathcal{T}(\mathcal{G}, \mathcal{H})$ , the edges leaving a node are all labeled differently, and, moreover, siblings are always differently labeled. Note, however, that different (non-sibling) nodes may have the same label, and so may edges originating from different nodes.

To give an example, consider Figure 3. Hypergraph vertices are denoted by letters, and hypergraph edges are denoted by numbers. In the tree illustrated, the root coincides with the pair of hypergraphs of Figure 1, except that the transversal  $\{d, b, f\}$  of  $\mathcal{G}$  is now missing in  $\mathcal{H}$ . The root is associated with the empty assignment  $\sigma_\varepsilon$ , and, correspondingly, the sets depicted with the root node are  $Sep(\sigma_\varepsilon)$ , and  $Com(\sigma_\varepsilon)$ . Each other node  $p$  represents an assignment  $\sigma_p$  whose included vertices are indicated by a checkmark (✓) and whose excluded vertices by a cross (X). In addition, each node  $p$  shows the separated edges of  $\mathcal{G}$  and the compatible edges of  $\mathcal{H}$  in  $\sigma_p$ , respectively. The left-most edge leaving the root is labeled with  $-a$  which stands for the exclusion of vertex  $a$ . This reflects the application of an extension-type (iii). On the other hand, the right-most edge leaving the root is labeled with  $(d, 1)$  which stands for the inclusion of vertex  $d$  as a critical vertex, along with edge 1 of  $\mathcal{G}$  witnessing  $d$ ’s criticality in the attempted new transversal under construction. This reflects the application of an extension-type (ii). In the given example, not all but only some nodes of the tree are depicted. Indeed, observe that the bottom right node of the figure is not a leaf, because its assignment is non-covering and still contains two free vertices that can be either included (as critical vertices), or excluded.

On the other hand, the bottom central node of the figure is a leaf, because its assignment is covering (in particular, edge 3 of hypergraph  $\mathcal{G}$  is covered by the excluded vertices).



A path  $\Pi = (\ell_1, \ell_2, \dots, \ell_k)$  in  $\mathcal{T}(\mathcal{G}, \mathcal{H})$  is a sequence of labels describing the path from the root to a node following the edges labeled in turn  $\ell_1, \ell_2, \dots, \ell_k$ .

For example, in Figure 3, the path  $\{(d, 1), -e\}$  leads to the bottom right node of the figure.

Since edges leaving a node are assumed to be all different, a path identifies unequivocally a node in the tree. Given a path  $\Pi$ , we denote by  $\mathcal{N}(\Pi)$  the end-node of  $\Pi$ .

The next Lemma, which shows how to compute the assignment  $\sigma_{\mathcal{N}(\Pi)}$  of the node  $\mathcal{N}(\Pi)$ , immediately follows from the definition of the concept of path. For notational convenience we define  $\sigma(\Pi) = \sigma_{\mathcal{N}(\Pi)}$ .

**Lemma 3.3.**

$$\sigma_{\mathcal{N}(\Pi)} = \sigma(\Pi) = \langle \bigcup_{(v,G) \in \Pi} \{v\}, (\bigcup_{-v \in \Pi} \{v\}) \cup (\bigcup_{(v,G) \in \Pi} (G \setminus \{v\})) \rangle. \quad (1)$$

Let us analyze what are the size reductions achieved when specific extension-types are performed. We denote by  $\varepsilon_v^{Sep(\sigma)} = \frac{|\{G \in Sep(\sigma) | v \in G\}|}{|Sep(\sigma)|}$  and  $\varepsilon_v^{Com(\sigma)} = \frac{|\{H \in Com(\sigma) | v \in H\}|}{|Com(\sigma)|}$  the ratio of the edges in  $Sep(\sigma)$  and  $Com(\sigma)$ , respectively, containing vertex  $v$ .

**Lemma 3.4.** *Let  $\mathcal{G}$  and  $\mathcal{H}$  be two hypergraphs satisfying the intersection property, and let  $\sigma$  be a non-covering assignment. If  $v$  is a free vertex of  $\sigma$ , then*

$$\begin{aligned} |Sep(\sigma + \langle \{v\}, \emptyset \rangle)| &= (1 - \varepsilon_v^{Sep(\sigma)}) \cdot |Sep(\sigma)|, \text{ and} \\ |Com(\sigma + \langle \emptyset, \{v\} \rangle)| &= (1 - \varepsilon_v^{Com(\sigma)}) \cdot |Com(\sigma)|. \end{aligned}$$

*On the other hand, if  $G \in Sep(\sigma)$ ,  $H \in Com(\sigma)$ ,  $v \in G$  and  $w \in H$  are free vertices of  $\sigma$ , then*

$$\begin{aligned} |Com(\sigma + \langle \{v\}, G \setminus \{v\} \rangle)| &\leq \varepsilon_v^{Com(\sigma)} \cdot |Com(\sigma)|, \text{ and} \\ |Sep(\sigma + \langle H \setminus \{w\}, \{w\} \rangle)| &\leq \varepsilon_w^{Sep(\sigma)} \cdot |Sep(\sigma)|. \end{aligned}$$

*Proof.* Given an assignment  $\sigma$ , if  $v$  is a free vertex in  $\sigma$  and  $v$  belongs to  $\varepsilon_v^{Sep(\sigma)} \cdot |Sep(\sigma)|$  many edges of  $Sep(\sigma)$ , then for the assignment  $\sigma' = \sigma + \langle \{v\}, \emptyset \rangle$  (extension-type (i)) it is easy to see that  $|Sep(\sigma')| = (1 - \varepsilon_v^{Sep(\sigma)}) \cdot |Sep(\sigma)|$ . Similarly, when the assignment  $\sigma' = \sigma + \langle \emptyset, \{v\} \rangle$  is considered (extension-type (iii)), it is easy to see that  $|Com(\sigma')| = (1 - \varepsilon_v^{Com(\sigma)}) \cdot |Com(\sigma)|$ .

Let us now consider extension-type (ii). Let  $\sigma' = \sigma + \langle \{v\}, G \setminus \{v\} \rangle$ , let  $K_{\{v\}} = \{\tilde{H} \in Com(\sigma) \mid \tilde{H} \cap G = \{v\}\}$ , and let  $K_{[v]} = \{\tilde{H} \in Com(\sigma) \mid \tilde{H} \cap G \ni v\}$ . Clearly  $K_{\{v\}} \subseteq K_{[v]}$ , and hence  $|K_{\{v\}}| \leq |K_{[v]}|$ . By definition,  $|K_{[v]}| = \varepsilon_v^{Com(\sigma)} \cdot |Com(\sigma)|$ , therefore  $|K_{\{v\}}| \leq \varepsilon_v^{Com(\sigma)} \cdot |Com(\sigma)|$ . Since  $\mathcal{G}$  and  $\mathcal{H}$  satisfy the intersection property, and  $Sep(\sigma)$  and  $Com(\sigma)$  contains edges of  $\mathcal{G}$  and  $\mathcal{H}$ , respectively, all the edges of  $Com(\sigma)$  intersect  $G$ . This, together with the fact that all the vertices  $G \setminus \{v\}$  are excluded in  $\sigma'$ , implies that  $Com(\sigma') = K_{\{v\}}$ . Therefore  $|Com(\sigma')| \leq \varepsilon_v^{Com(\sigma)} \cdot |Com(\sigma)|$ . For extension-type (iv) the proof is similar.  $\square$

Observe that in the proof of Lemma 3.4 we do not require  $Sep(\sigma)$  and  $Com(\sigma)$  to be simple. In fact, the lemma is valid regardless of that. Lemma 3.4 states a general property about the size reduction of  $Sep(\sigma)$  and  $Com(\sigma)$  when some assignments extensions are considered. Indeed, edges  $G$  from  $Sep(\sigma)$  do not need to be considered any longer as soon as they contain an included vertex, and this is irrespective of  $Sep(\sigma)$  being actually simple. A similar discussion extends to edges  $H$  of  $Com(\sigma)$  containing excluded vertices.

Before proceeding with our discussion, we recall that each node  $p$  of the tree corresponds to the sub-instance  $\mathcal{I}_p = \langle \mathcal{G}(\sigma_p), \mathcal{H}(\sigma_p) \rangle$  even though we do not explicitly represent it, and we refer instead to  $Sep(\sigma_p)$  and  $Com(\sigma_p)$ . Therefore  $\mathcal{T}(\mathcal{G}, \mathcal{H})$  is indeed a decomposition of the original instance into smaller sub-instances.

We claim that, by construction of the tree  $\mathcal{T}(\mathcal{G}, \mathcal{H})$ , if  $\mathcal{G}$  and  $\mathcal{H}$  are simple hypergraphs satisfying the intersection property, then the pair  $\langle \mathcal{G}, \mathcal{H} \rangle$  is a “yes”-instance of DUAL if and only if, for each node  $p$  of  $\mathcal{T}(\mathcal{G}, \mathcal{H})$ ,  $\mathcal{I}_p$  is a “yes”-instance of DUAL. Indeed, if  $\mathcal{G}$  and  $\mathcal{H}$  are dual, then, by Lemma 3.2, there is no assignment  $\sigma$  for which  $\mathcal{G}(\sigma)$  and  $\mathcal{H}(\sigma)$  are not dual, and hence all the nodes  $p$  of  $\mathcal{T}(\mathcal{G}, \mathcal{H})$  are such that  $\mathcal{I}_p$  is a “yes”-instance of DUAL. On the other hand, if  $\mathcal{G}$  and  $\mathcal{H}$  are not dual, since we assume they are simple and satisfy the intersection property, by Lemma 2.3 there exists a new transversal  $T$  of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$ . The fact that  $\mathcal{G}$  and  $\mathcal{H}$  are simple implies also that  $\mathcal{G} = \mathcal{G}(\sigma_\varepsilon) = \mathcal{G}(\sigma_r)$  and  $\mathcal{H} = \mathcal{H}(\sigma_\varepsilon) = \mathcal{H}(\sigma_r)$ , where  $r$  is the root of  $\mathcal{T}(\mathcal{G}, \mathcal{H})$ . Therefore, already the root  $r$  of  $\mathcal{T}(\mathcal{G}, \mathcal{H})$  is such that  $\mathcal{G}(\sigma_r)$  and  $\mathcal{H}(\sigma_r)$  are not dual, and hence  $\mathcal{I}_r$  is a “no”-instance of DUAL.

The critical point here is that, in order to prove that two hypergraphs  $\mathcal{G}$  and  $\mathcal{H}$  are actually not dual, it would be much better to identify in  $\mathcal{T}(\mathcal{G}, \mathcal{H})$  those nodes  $p$  for which it is computationally easy to verify that  $\mathcal{G}(\sigma_p)$  and  $\mathcal{H}(\sigma_p)$  are not dual, and the root of the tree may not always fit the purpose (of an efficient check). Therefore, to show that  $\langle \mathcal{G}, \mathcal{H} \rangle$  is a “no”-instance of DUAL, the ideal solution would be that of finding/guessing a path from the root to a node  $p$ , where  $\mathcal{I}_p$  is easily recognizable as a “no”-instance, e.g., as in the case in which  $\sigma_p = \langle In_p, Ex_p \rangle$  is such that the set  $In_p$  or  $Ex_p$  is a new transversal of  $\mathcal{G}$  or  $\mathcal{H}$ , respectively. The

interesting fact here is that, by construction of  $\mathcal{T}(\mathcal{G}, \mathcal{H})$ , if  $\mathcal{G}$  and  $\mathcal{H}$  are simple non-dual hypergraphs satisfying the intersection property, there is always a node  $p$  in  $\mathcal{T}(\mathcal{G}, \mathcal{H})$  such that  $\sigma_p$  has the required property for an easy check. Indeed, let us assume that  $T$  is a new transversal of  $\mathcal{G}$ , and consider the assignment  $\sigma = \langle \emptyset, \overline{T} \rangle$ . Since  $\sigma$  is an assignment which only excludes vertices, there exists a node  $p$  in  $\mathcal{T}(\mathcal{G}, \mathcal{H})$  such that  $\sigma_p = \sigma$  because we can build the assignment  $\sigma$  by successively using extensions of type (iii). It is clear that, in general, such a path may have a linear length. However, we will show below that if  $\langle \mathcal{G}, \mathcal{H} \rangle$  is indeed a “no”-instance of DUAL, then there must also exist a path of length  $O(\log N)$  to some node  $p$  such that checking that  $\mathcal{I}_p$  is a “no”-instance of DUAL is feasible within complexity class  $\text{TC}^0$ .

### 3.3 Logarithmic refuters in $\mathcal{T}(\mathcal{G}, \mathcal{H})$

An assignment  $\sigma = \langle In, Ex \rangle$  is said to be a *witness* of the existence of a new transversal of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$  if  $In$  is a new transversal of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$ , or  $Ex$  is a new transversal of  $\mathcal{H}$  w.r.t.  $\mathcal{G}$  (see Lemma 2.2). Similarly,  $\sigma$  is a *double witness* of the existence of a new transversal of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$  if  $In$  is a new transversal of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$ , and  $Ex$  is a new transversal of  $\mathcal{H}$  w.r.t.  $\mathcal{G}$ . Recall that a new transversal does not need to be minimal.

Note here that a node  $p$  of  $\mathcal{T}(\mathcal{G}, \mathcal{H})$  whose assignment  $\sigma_p$  is a witness is not necessarily a leaf of the tree. For example, in Figure 3 the assignment of the bottom right node is a witness, but this node, as already observed, is not a leaf of the full tree. From now on, we will often refer to properties of the assignment  $\sigma_p$  as properties of the node  $p$ . For example, we say that a node  $p$  of  $\mathcal{T}(\mathcal{G}, \mathcal{H})$  is a witness when  $\sigma_p$  is actually a witness.

We have already seen that, if  $\mathcal{G}$  and  $\mathcal{H}$  are simple non-dual hypergraphs satisfying the intersection property, then in  $\mathcal{T}(\mathcal{G}, \mathcal{H})$  there is always a witness at polynomial linear depth. But this is not enough for our purposes. Our aim in the rest of this section is to prove a stronger property. Indeed, we will show that there is always at only logarithmic depth a node that is either a witness or can be easily “extended” to be a witness.

Given a node  $p$  of  $\mathcal{T}(\mathcal{G}, \mathcal{H})$ , a free vertex  $v$  of  $\sigma_p$  is called *frequent* if  $v$  belongs to at least half of the edges in  $\text{Com}(\sigma_p)$ , otherwise we say that  $v$  is *infrequent*. For example, in Figure 3 vertex  $c$  is frequent at the root because it belongs to two out of four edges of  $\text{Com}(\sigma_\varepsilon) = \mathcal{H}$ . On the contrary, vertex  $d$  is infrequent at the root because it belongs to only one edge of  $\text{Com}(\sigma_\varepsilon) = \mathcal{H}$ . Let us denote by  $\text{Freq}_{\mathcal{G}, \mathcal{H}}(\sigma)$  and  $\text{Infreq}_{\mathcal{G}, \mathcal{H}}(\sigma)$  the vertices frequent and infrequent in  $\sigma$ , respectively. Again, we will often omit the subscript “ $\mathcal{G}, \mathcal{H}$ ” of  $\text{Freq}_{\mathcal{G}, \mathcal{H}}(\sigma)$  and  $\text{Infreq}_{\mathcal{G}, \mathcal{H}}(\sigma)$  when the hypergraphs  $\mathcal{G}$  and  $\mathcal{H}$  we are referring to are understood.

Assume that there exists a new transversal  $T$  of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$ . An assignment  $\sigma = \langle In, Ex \rangle$  is called a *precursor* of  $T$  if  $\sigma \sqsubseteq T$  and  $\sigma$  is *not* a witness. It is easy to see that, for such a precursor  $\sigma$ ,  $In \subset T$  and  $Ex \subset \overline{T}$ , for otherwise if  $In = T$  or  $Ex = \overline{T}$ , then  $\sigma$  would be a witness.

Before proceeding we need the following property.

**Lemma 3.5.** *Let  $\mathcal{G}$  and  $\mathcal{H}$  be two hypergraphs, and let  $\sigma = \langle In, Ex \rangle$  be a precursor of a new minimal transversal  $T$  of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$ . Then, every vertex  $v \in (T \setminus In)$  is free, and for each such vertex, there exists an edge  $G_v \in \text{Sep}(\sigma)$ , with  $v \in G_v$ , such that  $\sigma + \{\{v\}, G_v \setminus \{v\}\}$  is coherent with  $T$ .*

*Proof.* Let  $v$  be any vertex belonging to  $T \setminus In$ . From  $\sigma \sqsubseteq T$  it follows that  $v \notin Ex$  and hence  $v$  is free. Since  $T$  is a minimal transversal of  $\mathcal{G}$ , by Lemma 2.1,  $v$  is critical (in  $T$ ). For this reason, there exists an edge  $G_v \in \mathcal{G}$  such that  $T \cap G_v = \{v\}$ . Now, simply observe that  $G_v \in \text{Sep}(\sigma)$  and that  $\sigma + \{\{v\}, G_v \setminus \{v\}\} \sqsubseteq T$ .  $\square$

Let us consider a node  $p$  that is a precursor of a new *minimal* transversal  $T$  of  $\mathcal{G}$ . A free vertex  $v$  of  $\sigma_p$  is said to be *appealing to exclude* (for  $\sigma_p$ ) w.r.t.  $T$  if  $v \in \text{Freq}(\sigma_p)$  and  $v \notin T$ . On the other hand, a free vertex  $v$  of  $\sigma_p$  is said to be *appealing to include* (as a critical vertex) (for  $\sigma_p$ ) w.r.t.  $T$  if  $v \in \text{Infreq}(\sigma_p)$  and  $v \in T$ .

Consider again the example in Figure 3. Remember that the minimal transversal  $T = \{d, b, f\}$  of  $\mathcal{G}$  is missing in  $\mathcal{H}$ . At the root, vertex  $c$  is appealing for being excluded w.r.t.  $T$  because  $c$  is frequent and  $c \notin T$ . On the other hand, at the root, vertex  $d$  is appealing for being included as a critical vertex w.r.t.  $T$  (with edge 1 witnessing  $d$ ’s criticality) because  $d$  is infrequent and  $d \in T$ .

Observe that, given a node  $p$  precursor of a new minimal transversal  $T$ , edges leaving  $p$  labeled with the exclusion of an appealing vertex to exclude (w.r.t.  $T$ ), or labeled with the inclusion as a critical vertex (with a suitable criticality’s witness) of an appealing vertex to include (w.r.t.  $T$ ), lead to a node  $q$  such that  $\sigma_q \sqsubseteq T$  and  $|\text{Com}(\sigma_q)| \leq \frac{1}{2}|\text{Com}(\sigma_p)|$  (see Lemma 3.4). This is the key observation to prove that there exist in  $\mathcal{T}(\mathcal{G}, \mathcal{H})$  “duality refuters” at logarithmic depth.

We say that  $\sigma$  is a *saturated precursor* of a new minimal transversal  $T$  of  $\mathcal{G}$  when  $\sigma$  is a precursor of  $T$ , and no free vertex of  $\sigma$  is appealing to exclude or include for  $\sigma$  w.r.t.  $T$ . The next lemma, which states the most important property of saturated precursors, immediately follows from the concept of saturated precursor.

Given an assignment  $\sigma = \langle In, Ex \rangle$ ,  $\sigma^+ = \langle In \cup \text{Freq}(\sigma), Ex \cup \text{Infreq}(\sigma) \rangle$  is the *augmented assignment* of  $\sigma$ .

**Lemma 3.6.** *Let  $\mathcal{G}$  and  $\mathcal{H}$  be two hypergraphs, and let  $T$  be a new minimal transversal of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$ . If  $\sigma$  is a saturated precursor of  $T$ , then  $\sigma^+ = \langle T, \overline{T} \rangle$ , and thus  $\sigma^+$  is a double witness.*

We now prove the aforementioned crucial property of  $\mathcal{T}(\mathcal{G}, \mathcal{H})$ .

**Lemma 3.7.** *Let  $\mathcal{G}$  and  $\mathcal{H}$  be two hypergraphs. Then,  $\mathcal{G}$  and  $\mathcal{H}$  are not dual if and only if  $\mathcal{G}$  or  $\mathcal{H}$  is not simple, or  $\mathcal{G}$  and  $\mathcal{H}$  do not satisfy the intersection property, or there exists in  $\mathcal{T}(\mathcal{G}, \mathcal{H})$ , at depth at most  $\lfloor \log |\mathcal{H}| \rfloor + 1$ , a node  $p$  such that  $\sigma_p^+$  is a double witness.*

*Proof.*

( $\Rightarrow$ ) Assume that  $\mathcal{G}$  and  $\mathcal{H}$  are not dual. By Lemma 2.3,  $\mathcal{G}$  or  $\mathcal{H}$  is not simple, or they do not satisfy the intersection property, or there is a new transversal of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$ . If  $\mathcal{G}$  or  $\mathcal{H}$  is not simple, or they do not satisfy the intersection property, then this direction of the lemma trivially follows. So, let us assume that  $\mathcal{G}$  and  $\mathcal{H}$  are simple and satisfy the intersection property. We are going to show that there exists in  $\mathcal{T}(\mathcal{G}, \mathcal{H})$  a path  $\Pi$  of logarithmic length, starting from the root, such that  $\sigma(\Pi)^+$  is a double witness.

Since  $\mathcal{G}$  and  $\mathcal{H}$  are simple and satisfy the intersection property, by Lemma 2.3, there exists a new transversal of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$ . Let  $T$  be a new *minimal* transversal of  $\mathcal{G}$ , and  $p$  be a generic node of  $\mathcal{T}(\mathcal{G}, \mathcal{H})$  such that  $\sigma_p$  is a *non-saturated* precursor of  $T$  (i.e., a precursor of  $T$  that is *not* saturated). By Lemma 3.5, and by the fact that  $p$  is a non-saturated precursor of a minimal transversal, there is a child of  $p$ , say  $q$ , such that  $\sigma_q$  is coherent with  $T$ , and  $\sigma_q$  is obtained from  $\sigma_p$  through the inclusion (as a critical vertex) or the exclusion of an appealing vertex  $v$  for  $\sigma_p$  w.r.t.  $T$ . In particular, if  $v$  is appealing to include for  $\sigma_p$  w.r.t.  $T$ , then  $q$  is chosen by including  $v$  as a critical vertex, with an appropriate edge  $G_v \in \text{Sep}(\sigma_p)$ , such that  $G_v \cap T = \{v\}$ , witnessing the criticality of  $v$  in  $T$ . Otherwise, if  $v$  is appealing to exclude for  $\sigma_p$  w.r.t.  $T$ , then  $q$  is chosen by excluding  $v$ .

Observe that the empty assignment  $\sigma_\varepsilon$ , associated with the root of  $\mathcal{T}(\mathcal{G}, \mathcal{H})$ , is obviously a non-saturated precursor of  $T$ . Hence there exists a sequence of nodes  $s = (p_0, p_1, \dots, p_k)$ , where  $p_0$  is the root, such that all the nodes of  $s$  are coherent with  $T$ , and each node  $p_i$  is a child of  $p_{i-1}$  obtained through the inclusion (as a critical vertex) or the exclusion of an appealing vertex for  $\sigma_{p_{i-1}}$  w.r.t.  $T$ .

Let  $s$  be a maximum length sequence having the just mentioned property. Since  $s$  is of maximum length, node  $p_k$  is *not* a non-saturated precursor of  $T$  (for otherwise there would be a child of  $p_k$  allowing to extend  $s$ ). Hence, there are two cases: either (1)  $p_k$  is a saturated precursor of  $T$ , or (2)  $p_k$  is not a precursor of  $T$  at all.

For Case (1), by Lemma 3.6,  $\sigma_{p_k}^+$  is a double witness.

For Case (2), since  $\sigma_{p_k} = \langle \text{In}_{p_k}, \text{Ex}_{p_k} \rangle$  is coherent with  $T$ , if  $p_k$  is not a precursor of  $T$ , then  $p_k$  must be a witness (coherent with  $T$ ). Let  $F = V \setminus (\text{In}_{p_k} \cup \text{Ex}_{p_k})$  be the set of the free vertices of  $\sigma_{p_k}$ . Now, there are two cases: (a)  $\text{In}_{p_k}$  is a new transversal of  $\mathcal{G}$ ; or (b)  $\text{Ex}_{p_k}$  is a new transversal of  $\mathcal{H}$  (or both).

Let us first discuss Case (b) which is simpler. Observe that all the free vertices of  $\sigma_{p_k}$  are frequent because each of them belongs to at least half of the edges in  $\text{Com}(\sigma_{p_k})$  that, in this case, is empty because  $\text{Ex}_{p_k}$  is a transversal of  $\mathcal{H}$ . Hence the assignment  $\sigma_{p_k}^+ = \langle \text{In}_{p_k} \cup \text{Freq}(\sigma_{p_k}), \text{Ex}_{p_k} \cup \text{Infreq}(\sigma_{p_k}) \rangle = \langle \text{In}_{p_k} \cup F, \text{Ex}_{p_k} \rangle = \langle \overline{\text{Ex}_{p_k}}, \text{Ex}_{p_k} \rangle$  is a double witness (because  $\overline{\text{Ex}_{p_k}}$  is a new transversal of  $\mathcal{H}$ ; see Lemma 2.2).

Let us consider Case (a) in which we assume that  $\text{In}_{p_k}$  is a transversal of  $\mathcal{G}$ , and  $\text{Ex}_{p_k}$  is *not* a transversal of  $\mathcal{H}$ . From  $\sigma_{p_k} \subseteq T$  and  $T$  being a minimal transversal of  $\mathcal{G}$  it follows that  $\text{In}_{p_k} = T$ . Given that  $\text{In}_{p_k}$  is a new transversal of  $\mathcal{G}$ , it follows that for any subset  $A \subseteq F$  of the set of the free vertices of  $\sigma_{p_k}$ , the assignment  $\langle \text{In}_{p_k}, \text{Ex}_{p_k} \cup A \rangle$  is a witness (because  $\text{In}_{p_k}$  is a new transversal of  $\mathcal{G}$ ). Since excluding in  $\sigma_{p_k}$  vertices taken from  $F$  generates always witnessing assignments, we can extend the sequence  $s$  with a node of  $\mathcal{T}(\mathcal{G}, \mathcal{H})$  associated with the exclusion of a free and frequent vertex of  $\sigma_{p_k}$ . Furthermore, we can successively append to the new sequence other nodes of  $\mathcal{T}(\mathcal{G}, \mathcal{H})$  related to the exclusion of free and frequent vertices until the set of compatible edges of  $\mathcal{H}$  is empty or there are no more free frequent vertices.

Let  $s' = (p_0, p_1, \dots, p_k, p_{k+1}, \dots, p_r)$  be the longest amended sequence obtained from  $s$  in the just mentioned way. Let  $\sigma_{p_r} = \langle \text{In}_{p_r}, \text{Ex}_{p_r} \rangle$ , and let  $F' = V \setminus (\text{In}_{p_r} \cup \text{Ex}_{p_r})$  be the set of the free vertices of  $\sigma_{p_r}$ . By construction of  $s'$ ,  $\text{In}_{p_r} = \text{In}_{p_k}$  and hence  $\text{In}_{p_r} = T$  (because we are in Case (2)(a)). Now, there are two cases: either (I)  $\text{Com}(\sigma_{p_r}) = \emptyset$ ; or (II)  $\text{Com}(\sigma_{p_r}) \neq \emptyset$  (and there are no free frequent vertices in  $\sigma_{p_r}$ ).

Consider Case (I), which is similar to the Case (2)(b). Since  $\text{Com}(\sigma_{p_r}) = \emptyset$ , we claim that the assignment  $\sigma_{p_r}^+ = \langle \text{In}_{p_r} \cup \text{Freq}(\sigma_{p_r}), \text{Ex}_{p_r} \cup \text{Infreq}(\sigma_{p_r}) \rangle = \langle \text{In}_{p_r} \cup F', \text{Ex}_{p_r} \rangle = \langle \overline{\text{Ex}_{p_r}}, \text{Ex}_{p_r} \rangle$  is a double witness because  $\overline{\text{Ex}_{p_r}}$  is a new transversal of  $\mathcal{H}$ . Indeed,  $\overline{\text{Ex}_{p_r}}$  is a transversal of  $\mathcal{H}$  because  $\text{Com}(\sigma_{p_r}) = \emptyset$ . Moreover,  $\text{Ex}_{p_r}$  is an independent set of  $\mathcal{G}$  because  $\overline{\text{Ex}_{p_r}} = \text{In}_{p_r} \cup F'$  is a transversal of  $\mathcal{G}$  (since  $\text{In}_{p_r} = T$ ), and hence for every edge  $G \in \mathcal{G}$  there exists a vertex  $v \in G \cap \overline{\text{Ex}_{p_r}}$  which is equivalent to  $v \in G \setminus \text{Ex}_{p_r}$ .

Consider now Case (II). Since there are free vertices in  $\sigma_{p_r}$  because  $\text{Com}(\sigma_{p_r}) \neq \emptyset$ , and none of them is frequent, we claim that  $\sigma_{p_r}^+ = \langle \text{In}_{p_r} \cup \text{Freq}(\sigma_{p_r}), \text{Ex}_{p_r} \cup \text{Infreq}(\sigma_{p_r}) \rangle = \langle \text{In}_{p_r}, \text{Ex}_{p_r} \cup F' \rangle = \langle \text{In}_{p_r}, \overline{\text{In}_{p_r}} \rangle$  is a double witness. Indeed, since  $\text{In}_{p_r} = T$ ,  $\text{In}_{p_r}$  is a new transversal of  $\mathcal{G}$  and Lemma 2.2 applies.

Observe that  $s$  and  $s'$  have finite lengths, and hence they are well defined.

To conclude, by the definition of appealing vertex, it follows that  $|Com(\sigma_{p_i})| \leq \frac{1}{2}|Com(\sigma_{p_{i-1}})|$ , for all  $1 \leq i \leq k$  (see Lemma 3.4), and by the definition of  $s'$ , it follows that  $|Com(\sigma_{p_i})| \leq \frac{1}{2}|Com(\sigma_{p_{i-1}})|$ , for all  $k < i \leq r$ . Therefore  $s$  and  $s'$  each contain at most  $\lfloor \log |\mathcal{H}| \rfloor + 2$  nodes, and hence there exists a path  $\Pi$  of length at most  $\lfloor \log |\mathcal{H}| \rfloor + 1$  from the root to a node  $p$  such that  $\sigma_p^+$  is a witness.

( $\Leftarrow$ ) If  $\mathcal{G}$  or  $\mathcal{H}$  is not simple, or  $\mathcal{G}$  and  $\mathcal{H}$  do not satisfy the intersection property, then, by Lemma 2.3,  $\mathcal{G}$  and  $\mathcal{H}$  are not dual. Moreover, if there is, within the required depth, a node  $p$  such that  $\sigma_p^+$  is a double witness, then there exists a new transversal of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$ . Thus, by Lemma 2.3,  $\mathcal{G}$  and  $\mathcal{H}$  are not dual.  $\square$

## 4 New upper bounds for the DUAL problem

### 4.1 A new nondeterministic algorithm for DUAL

In this section we present our new nondeterministic algorithm ND-NOTDUAL for  $\overline{\text{DUAL}}$  and prove its correctness. Unlike previous algorithms, ND-NOTDUAL uses the novel data structure  $\mathcal{T}(\mathcal{G}, \mathcal{H})$  as defined in the previous section. To prove the correctness of the algorithm we will exploit the property of  $\mathcal{T}(\mathcal{G}, \mathcal{H})$  of having easily recognizable duality refuters at logarithmic depth (Lemma 3.7). Even though the characterization of that property was inspired by the algorithm of Gaur [24], it will become clear that our algorithm differs in essential aspects from Gaur's deterministic algorithm.

To disprove that two hypergraphs  $\mathcal{G}$  and  $\mathcal{H}$  are dual, we know that it is sufficient to show that at least one of them is not simple, or the intersection property does not hold between them, or there exists a new (minimal) transversal of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$  (see Lemma 2.3). Ruled out the first two conditions, intuitively, our nondeterministic algorithm, in order to compute such a new minimal transversal, guesses in the tree  $\mathcal{T}(\mathcal{G}, \mathcal{H})$  a path of logarithmic length leading to a node  $p$  such that  $\sigma_p^+$  is a double witness (see Lemma 3.7). More precisely, ND-NOTDUAL nondeterministically generates a set  $\Sigma$  of logarithmic-many labels, which is then checked to verify whether it is possible to derive from it a new transversal of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$ .

To be more formal, if  $\mathcal{G}$  and  $\mathcal{H}$  are two hypergraphs, for a node  $p$  of  $\mathcal{T}(\mathcal{G}, \mathcal{H}) = \langle N, A, r, \sigma, \ell \rangle$  let  $\mathcal{L}(p)$  denote the set of all the labels of the edges leaving  $p$ , i.e.,  $\mathcal{L}(p) = \{\ell((p, q)) : (p, q) \in A\}$ . By definition of  $\mathcal{T}(\mathcal{G}, \mathcal{H})$ , the labels of the edges leaving the root  $r$  are all the labels that may appear on any edge of  $\mathcal{T}(\mathcal{G}, \mathcal{H})$ , that is, for all  $a \in A$ ,  $\ell(a) \in \mathcal{L}(r)$ . A set of labels  $\Sigma$  of the tree  $\mathcal{T}(\mathcal{G}, \mathcal{H})$  is any subset of  $\mathcal{L}(r)$ . Note the difference between a path of  $\mathcal{T}(\mathcal{G}, \mathcal{H})$  and a set of labels of  $\mathcal{T}(\mathcal{G}, \mathcal{H})$ . The latter is just a set, while the former is an ordered sequence/list of labels coherent with the structure of  $\mathcal{T}(\mathcal{G}, \mathcal{H})$ . Given the above notation, we define the set

$$\mathcal{S}^{\log}(\mathcal{G}, \mathcal{H}) = \{\Sigma \mid \Sigma = \{\ell_1, \dots, \ell_k\} \text{ is a set of labels such that} \\ (0 \leq k \leq \lfloor \log |\mathcal{H}| \rfloor + 1) \wedge (\ell_i \in \mathcal{L}(r), \forall 1 \leq i \leq k)\}.$$

Given a set  $\Sigma \in \mathcal{S}^{\log}(\mathcal{G}, \mathcal{H})$ , the following expressions

$$\begin{aligned} In(\Sigma) &= \bigcup_{(v, G) \in \Sigma} \{v\} \\ Ex(\Sigma) &= \left( \bigcup_{-v \in \Sigma} \{v\} \right) \cup \left( \bigcup_{(v, G) \in \Sigma} (G \setminus \{v\}) \right) \end{aligned} \tag{2}$$

indicate the sets of the included and excluded vertices in  $\Sigma$ , respectively. These two expressions are similar to the formulas to compute an assignment given a (valid) path in  $\mathcal{T}(\mathcal{G}, \mathcal{H})$  (Formula (1) of Lemma 3.3). Since a sequence  $\Sigma$  is merely a set of labels, it may happen that  $In(\Sigma) \cap Ex(\Sigma) \neq \emptyset$ . When this is *not* the case we say that  $\Sigma$  is a *consistent* set of labels.

Given a set of labels  $\Sigma$ , we define  $\sigma(\Sigma)$  as the pair  $\langle In(\Sigma), Ex(\Sigma) \rangle$ . If  $\Sigma$  is consistent, then  $\sigma(\Sigma) = \langle In(\Sigma), Ex(\Sigma) \rangle$  is a (consistent) assignment, too. By a slight abuse of terminology and notation, given a set of labels  $\Sigma$ , regardless of whether  $\Sigma$  is actually consistent, we extend, in the natural way, the definitions given for (consistent) assignments to the pair  $\sigma(\Sigma) = \langle In(\Sigma), Ex(\Sigma) \rangle$ .

Given an assignment  $\sigma = \langle In, Ex \rangle$ , we define:

- $Mis_{\mathcal{G}, \mathcal{H}}(\sigma) = \{G \in \mathcal{G} \mid G \subseteq Ex\}$ , the set of all edges of  $\mathcal{G}$  *entirely missed* by  $\sigma$ ; and
- $Cov_{\mathcal{G}, \mathcal{H}}(\sigma) = \{H \in \mathcal{H} \mid H \subseteq In\}$ , the set of all edges of  $\mathcal{H}$  *entirely covered* by  $\sigma$ .<sup>6</sup>

<sup>6</sup>We could define a unique set of the edges of  $\mathcal{G}$  and  $\mathcal{H}$  covered by  $\sigma$ . We split them in two different sets to simplify the exposition in Section 4.2.

Again, we will often omit the subscript “ $\mathcal{G}, \mathcal{H}$ ” of  $Mis_{\mathcal{G}, \mathcal{H}}(\sigma)$  and  $Cov_{\mathcal{G}, \mathcal{H}}(\sigma)$  when the hypergraphs  $\mathcal{G}$  and  $\mathcal{H}$  we are referring to are understood. A witness is easily proven to be characterized as follows.

**Lemma 4.1.** *Let  $\mathcal{G}$  and  $\mathcal{H}$  be two hypergraphs. An assignment  $\sigma$  is a (non-duality) witness if and only if*

$$(Sep(\sigma) = \emptyset \wedge Cov(\sigma) = \emptyset) \vee (Com(\sigma) = \emptyset \wedge Mis(\sigma) = \emptyset). \quad (3)$$

We claim here that, given two hypergraphs  $\mathcal{G}$  and  $\mathcal{H}$ , checking Condition (3) of Lemma 4.1 is exactly what is needed to single out from  $\mathcal{S}^{\log}(\mathcal{G}, \mathcal{H})$  those sets of labels  $\Sigma$  which prove that  $\mathcal{G}$  and  $\mathcal{H}$  are not dual, regardless of  $\Sigma$  being actually (in)consistent. Indeed, by definition, the frequent and infrequent vertices of  $\sigma(\Sigma)$  are free vertices of  $\sigma(\Sigma)$ , and they are computed based on the frequencies with which they appear exclusively in the sets of  $Com(\sigma(\Sigma)) = \{H \in \mathcal{H} \mid H \cap Ex(\Sigma) = \emptyset\}$ . If  $\sigma(\Sigma)$  is actually inconsistent, the free vertices are simply those belonging neither to  $In(\Sigma)$  nor to  $Ex(\Sigma)$ . By this,  $Freq(\sigma(\Sigma))$  and  $Infreq(\sigma(\Sigma))$  are always a partition of the free vertices of  $\sigma(\Sigma)$ .

Let us consider  $\sigma(\Sigma)^+ = \langle A, B \rangle = \langle In(\Sigma) \cup Freq(\sigma(\Sigma)), Ex(\Sigma) \cup Infreq(\sigma(\Sigma)) \rangle$ . It is easy to see that, if  $\sigma(\Sigma)^+$  meets Condition (3) of Lemma 4.1, then  $A$  is a new transversal of  $\mathcal{G}$ , or  $B$  is a new transversal of  $\mathcal{H}$ , and hence  $\sigma(\Sigma)^+$  is a witness. This is because when the two disjuncts of Condition (3) are evaluated on  $\sigma(\Sigma)^+$ , they check whether  $A$  is a new transversal of  $\mathcal{G}$  and whether  $B$  is a new transversal of  $\mathcal{H}$ , respectively, and the two disjuncts are defined exclusively on the set  $A$  and  $B$ , respectively. Therefore, the fact that  $A$  and  $B$  are actually overlapping does not affect the soundness of the test.

**Lemma 4.2.** *Let  $\mathcal{G}$  and  $\mathcal{H}$  be two hypergraphs. Then,  $\mathcal{G}$  and  $\mathcal{H}$  are not dual if and only if  $\mathcal{G}$  or  $\mathcal{H}$  is not simple, or  $\mathcal{G}$  and  $\mathcal{H}$  do not satisfy the intersection property, or there exists a set  $\Sigma \in \mathcal{S}^{\log}(\mathcal{G}, \mathcal{H})$  such that  $\sigma(\Sigma)^+$  meets Condition (3) of Lemma 4.1.*

*Proof.*

- ( $\Rightarrow$ ) By Lemma 2.3, if  $\mathcal{G}$  and  $\mathcal{H}$  are not dual then  $\mathcal{G}$  or  $\mathcal{H}$  is not simple, or they do not satisfy the intersection property, or there exists a new transversal of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$ . If  $\mathcal{G}$  or  $\mathcal{H}$  is not simple, or they do not satisfy the intersection property, then this direction of the proof follows. Assume now that  $\mathcal{G}$  and  $\mathcal{H}$  are simple and satisfy the intersection property. By Lemma 3.7, if  $\mathcal{G}$  and  $\mathcal{H}$  are not dual, there exists a path  $\Pi$  of length at most  $\lceil \log |\mathcal{H}| \rceil + 1$  such that  $\sigma(\Pi)^+$  is a non-duality witness. Let  $\Sigma^\Pi$  be the set of labels containing exactly the labels of  $\Pi$ . Observe that  $\Sigma^\Pi$  actually belongs to  $\mathcal{S}^{\log}(\mathcal{G}, \mathcal{H})$ , and moreover  $\sigma(\Sigma^\Pi) = \sigma(\Pi)$ . Therefore, by Lemma 3.7,  $\sigma(\Sigma^\Pi)^+$  is a double witness and, by Lemma 4.1 meets Condition (3) of Lemma 4.1.
- ( $\Leftarrow$ ) By Lemma 2.3, if  $\mathcal{G}$  or  $\mathcal{H}$  is not simple, or they do not satisfy the intersection property, then  $\mathcal{G}$  and  $\mathcal{H}$  are not dual and this direction of the proof follows. Assume now that  $\mathcal{G}$  and  $\mathcal{H}$  are simple and satisfy the intersection property. If there exists a set of labels  $\Sigma \in \mathcal{S}^{\log}(\mathcal{G}, \mathcal{H})$  such that  $\sigma(\Sigma)^+$  meets Condition (3) of Lemma 4.1, then  $\sigma(\Sigma)^+$  is a witness and hence there exists a new transversal of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$  (remember that  $\sigma(\Sigma)$  being actually inconsistent does not affect the soundness of the test). Therefore, since  $\mathcal{G}$  and  $\mathcal{H}$  are simple and satisfy the intersection property, because there is a new transversal of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$ , by Lemma 2.3,  $\mathcal{G}$  and  $\mathcal{H}$  are not dual.  $\square$

We now present our nondeterministic algorithm. The pseudo-code of algorithm ND-NOTDUAL is listed as Algorithm 1; “**accept**” and “**reject**” are two commands causing a transition to a final accepting state and to a final rejecting state, respectively.

---

**Algorithm 1** A nondeterministic algorithm for  $\overline{\text{DUAL}}$ .

---

```

1: procedure ND-NOTDUAL( $\mathcal{G}, \mathcal{H}$ )
2:    $\Sigma \leftarrow \text{guess}(\text{a set of labels from } \mathcal{S}^{\log}(\mathcal{G}, \mathcal{H}));$ 
3:   if  $\neg \text{CHECK-SIMPLE-AND-INTERSECTION}(\mathcal{G}, \mathcal{H})$  then accept;
4:   if  $\text{CHECK-WITNESSAUGMENTED}(\mathcal{G}, \mathcal{H}, \Sigma)$  then accept;
5:   reject;

```

---

The two checking-procedures used in the algorithm implement the three deterministic tests needed after the guess has been carried out. The aims of the subprocedures are the following.

CHECK-SIMPLE-AND-INTERSECTION<sup>7</sup> checks whether the two hypergraphs  $\mathcal{G}$  and  $\mathcal{H}$  are simple, and the intersection property holds between them.

CHECK-WITNESSAUGMENTED checks whether the pair  $\sigma(\Sigma)^+$  witnesses the existence of a new transversal of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$  (or, equivalently, of  $\mathcal{H}$  w.r.t.  $\mathcal{G}$ ). In order to perform this check, Condition (3) of Lemma 4.1 is evaluated on  $\sigma(\Sigma)^+$ .

---

<sup>7</sup>This condition does not depend on the guessed sequence, and could thus be checked at the beginning of the algorithm, before the guess is made. However, for uniformity, and to adhere to a strict guess-and-check paradigm we check it after the guess.

**Theorem 4.3.** *Let  $\mathcal{G}$  and  $\mathcal{H}$  be two hypergraphs. Then, there is a computation branch of  $\text{ND-NOTDUAL}(\mathcal{G}, \mathcal{H})$  halting in an accepting state if and only if  $\mathcal{G}$  and  $\mathcal{H}$  are not dual.*

*Proof.*

( $\Rightarrow$ ) Let us assume that there exists a computation branch of  $\text{ND-NOTDUAL}(\mathcal{G}, \mathcal{H})$  halting in an accepting state. A transition to an accepting state may happen only at lines 3 or 4. If such a transition happens at line 3, then  $\mathcal{G}$  or  $\mathcal{H}$  is not simple, or  $\mathcal{G}$  and  $\mathcal{H}$  do not satisfy the intersection property. Hence, by Lemma 2.3,  $\mathcal{G}$  and  $\mathcal{H}$  are not dual. If a transition to the accepting state occurs at line 4, then it means that the set of labels  $\Sigma$  guessed in the current execution-branch is such that  $\sigma(\Sigma)^+$  is a witness. Therefore there exists a new transversal of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$ , and hence, since at this stage of the algorithm (line 4)  $\mathcal{G}$  and  $\mathcal{H}$  are guaranteed to be simple and satisfying the intersection property, by Lemma 2.3,  $\mathcal{G}$  and  $\mathcal{H}$  are not dual.

( $\Leftarrow$ ) Let us now assume that  $\mathcal{G}$  and  $\mathcal{H}$  are not dual. By Lemma 2.3,  $\mathcal{G}$  or  $\mathcal{H}$  is not simple, or they do not satisfy the intersection property, or there exists a new transversal of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$ .

If  $\mathcal{G}$  or  $\mathcal{H}$  is not simple, this condition is recognized by the algorithm at line 3 and the algorithm correctly moves to an accepting state. The same happens in case  $\mathcal{G}$  and  $\mathcal{H}$  do not satisfy the intersection property.

If  $\mathcal{G}$  and  $\mathcal{H}$  are simple and satisfy the intersection property, then, by Lemma 4.2, among the sets  $\Sigma$  guessed by the algorithm at line 2 there must be one such that  $\sigma(\Sigma)^+$  meets Condition (3) of Lemma 4.1. This condition is recognized by the algorithm at line 4 and the algorithm correctly moves to an accepting state.  $\square$

Note that our algorithm, while partly inspired by Gaur’s ideas, is fundamentally different from Gaur’s algorithm [24, 25]. In particular, Gaur’s algorithm may extend the set  $In$  of included vertices of an intermediate assignment  $\sigma = \langle In, Ex \rangle$  in a single step by several vertices and not by just one. In our approach this is only possible for end-nodes of the path. Moreover, algorithm  $\text{ND-NOTDUAL}$  could identify a witness by guessing a path of logarithmic length that is not a legal path according to Gaur because the single assignment-extensions are not chosen according to frequency counts. In fact, unlike Gaur’s algorithm,  $\text{ND-NOTDUAL}$  performs frequency counts only at the terminal nodes of a path.

## 4.2 Logical analysis of the $\text{ND-NOTDUAL}$ algorithm

We are going to show that, in order to disprove the duality of two hypergraphs  $\mathcal{G}$  and  $\mathcal{H}$ , the set  $\Sigma$  to which Lemma 4.2 refers can be recognized as meeting Condition (3) of Lemma 4.1 with (quite) low computational effort, and in particular within the complexity class  $\text{TC}^0$ . This will allow us to prove that  $\overline{\text{DUAL}} \in \text{GC}(\log^2 N, \text{TC}^0)$ . We begin by expressing the deterministic tests performed by  $\text{ND-NOTDUAL}$  in  $\text{FO}(\text{COUNT})$  which is first order logic augmented with the counting quantifiers “ $\exists!n$ ” having the following semantics.  $\text{FO}(\text{COUNT})$  is a two-sorted logic, i.e., a logic having two domain sets [35]: a numerical domain set containing objects used to interpret only numerical values, and another domain set containing all other objects. Consider the formula  $\Phi(n, x) = (\exists!n \ x)(\phi(x))$ . Variable  $x$  ranges over the non-numerical domain objects and is bound by the counting quantifier, while variable  $n$  ranges over the numerical domain objects and is left free by the counting quantifier.  $\Phi(n, x)$  is valid in all the interpretations in which  $n$  is substituted by the exact number of non-numerical domain values  $a$  for which  $\phi(a)$  evaluates to **true**.<sup>8</sup> Note that first order logic augmented with the majority quantifiers (FOM) is known to be equivalent to  $\text{FO}(\text{COUNT})$  [1, 35, 54]. The model checking problem for both logics is complete for the class  $\text{TC}^0$  [1, 35, 54].

With a pair of hypergraphs  $\langle \mathcal{G}, \mathcal{H} \rangle$  we associate a relational structure  $\mathcal{A}_{\langle \mathcal{G}, \mathcal{H} \rangle}$ . Essentially we represent hypergraphs through their incidence graphs. In particular, the universe  $A_{\langle \mathcal{G}, \mathcal{H} \rangle}$  of  $\mathcal{A}_{\langle \mathcal{G}, \mathcal{H} \rangle}$  consists of an object for each vertex of  $V$ , an object for each hyperedge of the two hypergraphs, and two more objects,  $o_{\mathcal{G}}$  and  $o_{\mathcal{H}}$ , for the two hypergraphs, i.e.,  $A_{\langle \mathcal{G}, \mathcal{H} \rangle} = \{o_v \mid v \in V\} \cup \{o_G \mid G \in \mathcal{G}\} \cup \{o_H \mid H \in \mathcal{H}\} \cup \{o_{\mathcal{G}}, o_{\mathcal{H}}\}$ .

The relations of  $\mathcal{A}_{\langle \mathcal{G}, \mathcal{H} \rangle}$  are as follows:  $\text{Vertex}(x)$  is a unary relation indicating that object  $x$  is a vertex;  $\text{Hyp}(x)$  is a unary relation indicating that object  $x$  is a hypergraph;  $\text{EdgeOf}(x, y)$  is a binary relation indicating that object  $x$  is an edge of the hypergraph identified by object  $y$ ; and  $\text{In}(x, y)$  is the binary incidence relation indicating that object  $x$  is a vertex belonging to the edge identified by object  $y$ .

We also need to represent through relations the guessed set  $\Sigma$ . Remember that in  $\Sigma$  there are elements of two types:  $-v$  where  $v$  is a vertex, and  $(v, G)$  where  $v$  is a vertex and  $G$  is an edge of  $\mathcal{G}$ .<sup>9</sup> We assume a unary relation  $S_1$  storing those tuples  $\langle v \rangle$  where  $v$  is a vertex such that  $-v \in \Sigma$ , moreover we assume a binary relation  $S_2$  containing those tuples  $\langle v, G \rangle$  where  $v$  is a vertex and  $G$  is an edge such that  $(v, G) \in \Sigma$ .

<sup>8</sup>For more on this, the reader is referred to any standard textbook on the topic. See, e.g., [10, 11, 35, 45].

<sup>9</sup>Note that an edge  $G$  in a label of a path or a set is given by its identifier and not by the explicit list of its vertices.

Remember that, by Lemma 4.2, it is sufficient to guess a set of labels, and it is not required to guess a path. This means that the exact order of the labels is not relevant, and hence the above relational representation of a guessed set is totally sufficient.

We use the following “macros” in our first order formulas:

$$\begin{aligned} v \in V &\equiv \text{Vertex}(v) \\ g \in \mathcal{G} &\equiv \text{Hyp}(o_{\mathcal{G}}) \wedge \text{EdgeOf}(g, o_{\mathcal{G}}) \\ h \in \mathcal{H} &\equiv \text{Hyp}(o_{\mathcal{H}}) \wedge \text{EdgeOf}(h, o_{\mathcal{H}}) \\ v \in g &\equiv \text{In}(v, g) \end{aligned}$$

We are now ready to prove some intermediate results.

**Lemma 4.4.** *Let  $\mathcal{G}$  be a hypergraph. Deciding whether  $\mathcal{G}$  is simple is expressible in FO.*

*Proof.* We know that a hypergraph  $\mathcal{G}$  is simple if and only if, for all pairs of distinct edges  $G, H \in \mathcal{G}$ ,  $G \not\subseteq H$ . Hence, the formula checking whether a hypergraph is simple is:

$$\text{simple}(x) \equiv \text{Hyp}(x) \wedge (\forall g, h)((g \in x \wedge h \in x \wedge g \neq h) \rightarrow (\exists v)(v \in V \wedge v \in g \wedge \neg(v \in h))). \quad \square$$

**Lemma 4.5.** *Let  $\mathcal{G}$  and  $\mathcal{H}$  be two hypergraphs. Deciding whether  $\mathcal{G}$  and  $\mathcal{H}$  satisfy the intersection property is expressible in FO.*

*Proof.* Two hypergraphs  $\mathcal{G}$  and  $\mathcal{H}$  satisfy the intersection property if and only if, for every pair of edges  $G \in \mathcal{G}$  and  $H \in \mathcal{H}$ ,  $G \cap H \neq \emptyset$ . Hence, the formula encoding this test is:

$$\text{intersection-property} \equiv (\forall g, h)((g \in \mathcal{G} \wedge h \in \mathcal{H}) \rightarrow (\exists v)(v \in V \wedge v \in g \wedge v \in h)). \quad \square$$

We say that the guess is congruent if for every guessed tuple  $\langle x \rangle \in S_1$  the object  $x$  is actually a vertex, and for every tuple  $\langle x, y \rangle \in S_2$  the object  $y$  is actually an edge belonging to  $\mathcal{G}$  containing the vertex identified by the object  $x$ .

**Lemma 4.6.** *Let  $\mathcal{G}$  and  $\mathcal{H}$  be two hypergraphs, and let  $\Sigma$  be a (guessed) set of labels of  $\mathcal{T}(\mathcal{G}, \mathcal{H})$ . Deciding the congruency of  $\Sigma$  is expressible in FO.*

*Proof.* The congruency of the guessed set can be checked through:

$$\text{congruentGuess} \equiv (\forall v)(S_1(v) \rightarrow v \in V) \wedge (\forall w, g)(S_2(w, g) \rightarrow w \in V \wedge g \in \mathcal{G} \wedge w \in g). \quad \square$$

To conclude our complexity analysis of the deterministic tests performed by ND-NOTDUAL, let us formulate in FO(COUNT) the property that  $\sigma(\Sigma)^+$  meets Condition (3) of Lemma 4.1.

**Lemma 4.7.** *Let  $\mathcal{G}$  and  $\mathcal{H}$  be two hypergraphs, and let  $\Sigma$  be a (guessed) set of labels of  $\mathcal{T}(\mathcal{G}, \mathcal{H})$ . Deciding whether  $\sigma(\Sigma)^+$  meets Condition (3) of Lemma 4.1 is expressible in FO(COUNT).*

*Proof.* Let  $\sigma(\Sigma) = \langle \text{In}(\Sigma), \text{Ex}(\Sigma) \rangle$  be the pair associated with  $\Sigma$ . Essentially we need to prove that is possible to express in FO(COUNT) Condition (3) of Lemma 4.1 on  $\sigma(\Sigma)^+$ . Remember that  $\sigma(\Sigma)$  is not explicitly represented, but it can be evaluated from  $\Sigma$  through Formulas (2) of Section 4.1.

Let us define the following two formulas serving the purpose to evaluate if a vertex belongs to  $\text{In}(\Sigma)$  or  $\text{Ex}(\Sigma)$ , respectively.

$$\begin{aligned} I\text{-guess}(v) &\equiv v \in V \wedge (\exists g)(S_2(v, g)) \\ E\text{-guess}(v) &\equiv v \in V \wedge (S_1(v) \vee (\exists w, g)(S_2(w, g) \wedge w \neq v \wedge v \in g)). \end{aligned}$$

We now exhibit the formulas to verify whether a given vertex is frequent in  $\sigma(\Sigma)$ . These formulas are the only ones in which we actually use the counting quantifier. In the following formulas we will use predicate  $PLUS(x, y, z)$ , which holds **true** whenever  $x + y = z$ , and predicate  $SUCC(x, y)$ , which holds **true** whenever  $x$  and  $y$  are two values of the domain such that  $y$  is the immediate successor of  $x$  in the ordering of the domain. Remember, indeed, that the relational structure is assumed to have a totally ordered domain (and the predicate  $<$  allow us to test the ordering), and to have a predicate  $BIT(i, j)$  that holds **true** whenever the  $j^{\text{th}}$  bit of the binary representation of number  $i$  is 1. These assumptions allow to express in first-order logic the predicates  $PLUS(x, y, z)$  and  $SUCC(x, y)$  (see Section 1.2 of [35]).

Since we need to evaluate whether a vertex  $v$  is frequent in  $\sigma(\Sigma)$ , we have to check whether  $v$  belongs to at least  $\lceil |Com(\sigma(\Sigma))|/2 \rceil$  edges of  $Com(\sigma(\Sigma))$ . So we exhibit a formula  $half(x, y)$  which holds **true** whenever  $y = \lceil x/2 \rceil$ :

$$half(x, y) \equiv PLUS(y, y, x) \vee (\exists z)(PLUS(y, y, z) \wedge SUCC(x, z)).$$

The following formulas evaluate respectively: whether an edge belongs to  $Com(\sigma(\Sigma))$ , the number of edges in  $Com(\sigma(\Sigma))$ , the number of edges in  $Com(\sigma(\Sigma))$  containing a given vertex  $v$ , and whether  $v$  is frequent in  $\sigma(\Sigma)$ .

$$\begin{aligned}
com(h) &\equiv h \in \mathcal{H} \wedge (\forall v)((v \in V \wedge v \in h) \rightarrow \neg E\text{-}guess(v)) \\
count\text{-}com\text{-}\mathcal{H}(n) &\equiv (\exists!n \ h)(h \in \mathcal{H} \wedge com(h)) \\
count\text{-}com\text{-}\mathcal{H}\text{-}inc(v, n) &\equiv v \in V \wedge (\exists!n \ h)(h \in \mathcal{H} \wedge com(h) \wedge v \in h) \\
freq(v) &\equiv v \in V \wedge \neg I\text{-}guess(v) \wedge \neg E\text{-}guess(v) \wedge \\
&(\exists n, m, o)(count\text{-}com\text{-}\mathcal{H}(n) \wedge count\text{-}com\text{-}\mathcal{H}\text{-}inc(v, m) \wedge half(n, o) \wedge (o = m \vee o < m)).
\end{aligned}$$

After having defined a formula to evaluate whether a vertex is frequent in  $\sigma(\Sigma)$ , we show the formulas computing the included and excluded vertices of the augmented pair  $\sigma(\Sigma)^+$ , respectively:

$$\begin{aligned}
I\text{-}aug(v) &\equiv v \in V \wedge (I\text{-}guess(v) \vee freq(v)) \\
E\text{-}aug(v) &\equiv v \in V \wedge (E\text{-}guess(v) \vee \neg freq(v)).
\end{aligned}$$

Now we exhibit the formulas encoding the evaluation of Condition (3) of Lemma 4.1 on  $\sigma(\Sigma)^+$ . The formulas evaluating whether an edge belongs to  $Mis(\sigma(\Sigma)^+)$ , to  $Cov(\sigma(\Sigma)^+)$ , to  $Sep(\sigma(\Sigma)^+)$ , and to  $Com(\sigma(\Sigma)^+)$ , are, respectively:

$$\begin{aligned}
mis\text{-}aug(g) &\equiv g \in \mathcal{G} \wedge (\forall v)((v \in V \wedge v \in g) \rightarrow E\text{-}aug(v)) \\
cov\text{-}aug(h) &\equiv h \in \mathcal{H} \wedge (\forall v)((v \in V \wedge v \in h) \rightarrow I\text{-}aug(v)) \\
sep\text{-}aug(g) &\equiv g \in \mathcal{G} \wedge (\forall v)((v \in V \wedge v \in g) \rightarrow \neg I\text{-}aug(v)) \\
com\text{-}aug(h) &\equiv h \in \mathcal{H} \wedge (\forall v)((v \in V \wedge v \in h) \rightarrow \neg E\text{-}aug(v)).
\end{aligned}$$

Finally the formula verifying that  $\sigma(\Sigma)^+$  meets Condition (3) of Lemma 4.1 is as follows.

$$\begin{aligned}
guessAugWitness &\equiv ((\forall g)(g \in \mathcal{G} \rightarrow \neg sep\text{-}aug(g)) \wedge (\forall h)(h \in \mathcal{H} \rightarrow \neg cov\text{-}aug(h))) \vee \\
&((\forall h)(h \in \mathcal{H} \rightarrow \neg com\text{-}aug(h)) \wedge (\forall g)(g \in \mathcal{G} \rightarrow \neg mis\text{-}aug(g))).
\end{aligned}$$

□

### 4.3 Putting it all together

We are now ready to prove our main results.

**Theorem 4.8.** *Let  $\mathcal{G}$  and  $\mathcal{H}$  be two hypergraphs. Deciding whether  $\mathcal{G}$  and  $\mathcal{H}$  are not dual is feasible in  $GC(\log^2 N, TC^0)$ .*

*Proof.* Lemmas 4.4, 4.5, 4.6, and 4.7 show that the deterministic checks performed by the algorithm ND-NOTDUAL are expressible in  $FO(COUNT)$ . Hence these tests are feasible in logtime-uniform  $TC^0$  [1, 35, 54].

Moreover, by analyzing the algorithm ND-NOTDUAL, it is evident that only  $O(\log^2 N)$  nondeterministic bits are sufficient to be guessed. Indeed, if  $\mathcal{G}$  or  $\mathcal{H}$  is not simple, or  $\mathcal{G}$  and  $\mathcal{H}$  do not satisfy the intersection property, then the guessed set  $\Sigma$  is completely ignored, because  $\mathcal{G}$  and  $\mathcal{H}$  are directly recognized not to be dual, and hence is totally irrelevant what the guessed bits are. On the other hand, if  $\mathcal{G}$  and  $\mathcal{H}$  are simple and satisfy the intersection property, then, by Lemma 4.2, there exists in  $\mathcal{S}^{\log}(\mathcal{G}, \mathcal{H})$  a set  $\Sigma$  with  $O(\log |\mathcal{H}|)$  elements such that  $\sigma(\Sigma)^+$  meets Condition (3) of Lemma 4.1 if and only if  $\mathcal{G}$  and  $\mathcal{H}$  are not dual. Remember that our definition of the size of the input of DUAL is  $N = \|\mathcal{G}\| + \|\mathcal{H}\|$ , hence the number of elements of  $\Sigma$  is also  $O(\log N)$ . Since, by definition,  $O(\log N)$  bits are sufficient to represent any vertex or edge ID of the input hypergraphs, each label of  $\Sigma$  can be represented with only  $O(\log N)$  bits. By this, the whole set  $\Sigma$  can be correctly represented and stored in the (set) variable  $\Sigma$  with  $O(\log^2 N)$  bits.

Therefore,  $\overline{DUAL}$  belongs to  $GC(\log^2 N, TC^0)$ . □

From the previous theorem the following corollaries follows immediately, the first of which proves that the conjecture stated by Gottlob [28] actually holds.

**Corollary 4.9.** *Let  $\mathcal{G}$  and  $\mathcal{H}$  be two hypergraphs. Deciding whether  $\mathcal{G}$  and  $\mathcal{H}$  are not dual is feasible in  $GC(\log^2 N, LOGSPACE)$ .*

*Proof.* Follows immediately from Theorem 4.8 and the well-known inclusion  $TC^0 \subseteq LOGSPACE$ . □

**Corollary 4.10** ([28]). *Let  $\mathcal{G}$  and  $\mathcal{H}$  be two hypergraphs. Deciding whether  $\mathcal{G}$  and  $\mathcal{H}$  are dual is feasible in  $DSPACE[\log^2 N]$ .*

*Proof.* By the inclusion  $GC(\log^2 N, LOGSPACE) \subseteq DSPACE[\log^2 N]$ , from Corollary 4.9 follows  $\overline{DUAL} \in DSPACE[\log^2 N]$ . Since  $DSPACE[\log^2 N]$  is closed under complement,  $DUAL \in DSPACE[\log^2 N]$ . □



## 4.4 Computing a new transversal

In this section, we will show that computing a (not necessarily minimal) new transversal of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$  is feasible in space  $O(\log^2 N)$ . At first, observe that it is possible to define a total order  $\preceq$  over  $\mathcal{S}^{\log}(\mathcal{G}, \mathcal{H})$ . Indeed, consider the totally ordered domain  $A_{\langle \mathcal{G}, \mathcal{H} \rangle}$  of the structure  $\mathcal{A}_{\langle \mathcal{G}, \mathcal{H} \rangle}$ , and in particular consider the space of the pairs  $P = A_{\langle \mathcal{G}, \mathcal{H} \rangle} \times A_{\langle \mathcal{G}, \mathcal{H} \rangle}$ . Let us at first define an order over  $P$ : Given two pairs  $p_1 = \langle a_1, b_1 \rangle$  and  $p_2 = \langle a_2, b_2 \rangle$  belonging to  $P$ ,  $p_1$  precedes  $p_2$  in  $P$  if and only if  $(a_1 < a_2) \vee (a_1 = a_2 \wedge b_1 < b_2)$ , where  $<$  is the ordering relation over  $A_{\langle \mathcal{G}, \mathcal{H} \rangle}$ .

Now, we associate each label with a pair in  $P$ : the label  $(v, G)$  is associated with the pair  $\langle o_v, o_G \rangle \in P$ , where  $o_v$  and  $o_G$  are the objects of  $A_{\langle \mathcal{G}, \mathcal{H} \rangle}$  associated with  $v$  and  $G$ , respectively; and  $-v$  is associated with the pair  $\langle o_v, o_v \rangle$ , where  $o_v$  is the object of  $A_{\langle \mathcal{G}, \mathcal{H} \rangle}$  associated with  $v$ . Given two labels  $\ell_1$  and  $\ell_2$  and their respective associated pairs  $p_1$  and  $p_2$ ,  $\ell_1$  precedes  $\ell_2$  if and only if  $p_1$  precedes  $p_2$  in  $P$ .

To conclude, given two sets of labels  $\Sigma_1, \Sigma_2 \in \mathcal{S}^{\log}(\mathcal{G}, \mathcal{H})$ ,  $\Sigma_1$  precedes  $\Sigma_2$ , denoted by  $\Sigma_1 \prec \Sigma_2$ , if and only if  $\Sigma_1$  contains strictly less labels than  $\Sigma_2$ , or  $\Sigma_1$  and  $\Sigma_2$  contain the same number of labels and the least labels  $\ell_1 \in \Sigma_1$  and  $\ell_2 \in \Sigma_2$  on which  $\Sigma_1$  and  $\Sigma_2$  differ are such that  $\ell_1$  precedes  $\ell_2$ .

Given this order it is possible to enumerate all the sets belonging to  $\mathcal{S}^{\log}(\mathcal{G}, \mathcal{H})$  without repetitions.

Consider now the following *deterministic* algorithm COMPUTENT listed as Algorithm 2, which, given two hypergraphs  $\mathcal{G}$  and  $\mathcal{H}$ , successively generates all the sets  $\Sigma$  belonging to  $\mathcal{S}^{\log}(\mathcal{G}, \mathcal{H})$  to see if one of them is a good starting point to build a new transversal of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$ . A prerequisite for the correct execution of the algorithm COMPUTENT is that the input hypergraphs satisfy the intersection property, and the purpose of the procedure CHECK-INTERSECTIONPROPERTY used in COMPUTENT is precisely that. This is required because COMPUTENT looks for sets of labels only among those in  $\mathcal{S}^{\log}(\mathcal{G}, \mathcal{H})$ . In the pseudo-code of the algorithm “**error**” is a command triggering an error state/signal.

---

**Algorithm 2** A deterministic algorithm, derived from ND-NOTDUAL, computing a new transversal of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$ . Here neither  $\sigma(\Sigma)$  nor  $\sigma(\Sigma)^+$  are explicitly stored, but they are dynamically computed as needed. We assume that  $\sigma(\Sigma) = \langle In(\Sigma), Ex(\Sigma) \rangle$ .

---

### Require:

Hypergraphs  $\mathcal{G}$  and  $\mathcal{H}$  satisfy the intersection property.

- 1: **procedure** COMPUTENT( $\mathcal{G}, \mathcal{H}$ )
  - 2:   **if**  $\neg$ CHECK-INTERSECTIONPROPERTY( $\mathcal{G}, \mathcal{H}$ ) **then error**;
  - 3:   **for** each  $\Sigma$ :  $\Sigma \in \mathcal{S}^{\log}(\mathcal{G}, \mathcal{H})$  **do**
  - 4:     **if**  $(Sep(\sigma(\Sigma)^+) = \emptyset \wedge Cov(\sigma(\Sigma)^+) = \emptyset)$  **then output**  $In(\Sigma) \cup Freq(\sigma(\Sigma))$ ;
  - 5:     **else if**  $(Com(\sigma(\Sigma)^+) = \emptyset \wedge Mis(\sigma(\Sigma)^+) = \emptyset)$  **then output**  $V \setminus (Ex(\Sigma) \cup Infreq(\sigma(\Sigma)))$ ;
  - 6:   **output NIL**;
- 

**Lemma 4.11.** *Let  $\mathcal{G}$  and  $\mathcal{H}$  be two hypergraphs satisfying the intersection property. The algorithm COMPUTENT correctly computes a new transversal of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$  (if it exists) in space  $O(\log^2 N)$ .*

*Proof.* COMPUTENT always terminates because all the sets belonging to  $\mathcal{S}^{\log}(\mathcal{G}, \mathcal{H})$  are finite and, by exploiting the order defined, can be enumerated successively without repetitions.

At first, COMPUTENT checks the intersection property, and if this property does not hold between the two input hypergraphs an error state/signal is triggered. Then COMPUTENT successively enumerates all the possible elements belonging to  $\mathcal{S}^{\log}(\mathcal{G}, \mathcal{H})$  to find (if one exists) a set  $\Sigma$  such that  $\sigma(\Sigma)^+$  meets Condition (3) of Lemma 4.1. Now, there are two cases: either (a)  $\mathcal{G}$  and  $\mathcal{H}$  are not dual, or (b) they are.

Let us consider Case (a). At this stage of the algorithm (i.e., after line 2), hypergraphs  $\mathcal{G}$  and  $\mathcal{H}$  are guaranteed to satisfy the intersection property. Hence, by Lemma 2.3, since  $\mathcal{G}$  and  $\mathcal{H}$  are not dual,  $\mathcal{G}$  or  $\mathcal{H}$  is not simple, or there exists a new transversal of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$ . Observe that, if  $\mathcal{G}$  or  $\mathcal{H}$  is not simple this does not affect the existence of a new transversal of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$ . Indeed, it can be easily shown that there exists a new transversal of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$  if and only if there exists a new transversal of  $\min(\mathcal{G})$  w.r.t.  $\min(\mathcal{H})$  (i.e., we eliminate those edges containing other edges). By this observation and Lemmas 4.2 and 4.1, COMPUTENT, at line 3, runs across a set  $\Sigma$  such that  $\sigma(\Sigma)^+$  meets Condition (3) of Lemma 4.1 if and only if there exists a new transversal of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$ .

Let  $\sigma(\Sigma)^+ = \langle A, B \rangle = \langle In(\Sigma) \cup Freq(\sigma(\Sigma)), Ex(\Sigma) \cup Infreq(\sigma(\Sigma)) \rangle$ . Observe that, at lines 4 and 5 of COMPUTENT it is checked whether  $A$  is a new transversal of  $\mathcal{G}$ , and whether  $B$  is a new transversal of  $\mathcal{H}$ , respectively. We know that if those tests return **true**, then  $A$  and  $B$  are actually new transversals of  $\mathcal{G}$  and  $\mathcal{H}$ , respectively (again, remember that  $\sigma(\Sigma)$  being actually inconsistent does not affect the soundness of the test). If the test at line 4 is passed, then the set  $A$  is output, because  $A$  is a new transversal of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$ . On the

other hand, if the test at line 5 is passed, then the set  $\overline{B}$ , is output, because  $B$  is a new transversal of  $\mathcal{H}$ , and hence, by Lemma 2.2,  $\overline{B}$  is a new transversal of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$ .

Consider now Case (b), in which  $\mathcal{G}$  and  $\mathcal{H}$  are dual. By Lemma 4.2, if  $\mathcal{G}$  and  $\mathcal{H}$  are dual, then there is no set of labels belonging to  $\mathcal{S}^{\log}(\mathcal{G}, \mathcal{H})$  that passes any of the two tests at lines 4 and 5. In this case, the algorithm COMPUTENT correctly outputs a null (*NIL*) result, at line 6, because no new transversal of  $\mathcal{G}$  exists.

To conclude, let us show that COMPUTENT executes within a quadratic logspace bound. All the sets  $\Sigma$  generated contain at most  $\lfloor \log |\mathcal{H}| \rfloor + 1$  labels, which is  $O(\log N)$ , and each of these sets can be represented with  $O(\log^2 N)$  bits. For this reason, by re-using of work-space, the algorithm needs only  $O(\log^2 N)$  bits to represent all the sets successively tried. Lemmas 4.5 and 4.7 show that all the tests can be executed in  $\text{TC}^0$  and hence in logarithmic space (by the inclusion  $\text{TC}^0 \subseteq \text{LOGSPACE}$ ). In fact, in order to implement those tests within a logarithmic space bound, the pairs  $\sigma(\Sigma)$  and  $\sigma(\Sigma)^+$ , the sets of the separated, missed, compatible, and covered edges, and the sets of frequent and infrequent vertices, are dynamically computed in LOGSPACE when needed, rather than being explicitly stored.

Observe also that the output operations can be carried out in logarithmic space. Indeed, the elements belonging to  $A = \text{In}(\Sigma) \cup \text{Freq}(\sigma(\Sigma))$ , and  $\overline{B} = V \setminus (\text{Ex}(\Sigma) \cup \text{Infreq}(\sigma(\Sigma)))$ , can be output successively one by one using only logarithmic workspace by exploiting Formulas (2) of Section 4.1 (for each vertex  $v$  it is decided whether  $v$  must be output or not). Note that given a vertex  $v$ , checking whether  $v$  is a free vertex of  $\sigma(\Sigma)$  is feasible in  $\text{TC}^0$  (see the proof of Lemma 4.7), and hence in LOGSPACE (from  $\text{TC}^0 \subseteq \text{LOGSPACE}$ ). Moreover, deciding whether a free vertex of  $\sigma(\Sigma)$  is frequent is feasible in  $\text{TC}^0$  (see the proof of Lemma 4.7), and hence in LOGSPACE.  $\square$

It is an open problem whether it is possible to compute a *minimal* new transversal of  $\mathcal{G}$  in space  $O(\log^2 N)$ .

From the previous lemma, the following theorem directly follows. Note that the result here reported is actually a (slight) improvement over the result in the conference paper [28] because we require here that the input hypergraphs satisfy the intersection property instead of the tighter condition of  $\mathcal{G}$  and  $\mathcal{H}$  being such that  $\mathcal{G} \subseteq \text{tr}(\mathcal{H})$  and  $\mathcal{H} \subseteq \text{tr}(\mathcal{G})$ .

**Theorem 4.12** (improved over [28]). *Let  $\mathcal{G}$  and  $\mathcal{H}$  be two hypergraphs satisfying the intersection property. Then, computing a new (not necessarily minimal) transversal of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$  is feasible in  $\text{FDSPACE}[\log^2 N]$ .*

## 5 Conclusions and future research

In this paper, we studied the computational complexity of the DUAL problem. By using standard decomposition techniques for DUAL, we proved that after logarithmic many decomposition steps it is possible to individuate a sub-instance of the original instance of DUAL, for which verifying that it corresponds to a new transversal is feasible within complexity class  $\text{TC}^0$ .

From this we devised a new nondeterministic algorithm for  $\overline{\text{DUAL}}$  whose analysis allowed us to recognize  $\text{GC}(\log^2 n, \text{TC}^0)$  as a new complexity upper bound for  $\overline{\text{DUAL}}$ . As a simple corollary of this results, we obtained also that  $\overline{\text{DUAL}} \in \text{GC}(\log^2 n, \text{LOGSPACE})$ , which was conjectured by Gottlob [28].

Then the nondeterministic algorithm proposed in this paper is used to develop a simple deterministic algorithm whose space complexity is  $O(\log^2 n)$ .

Now some questions arise. Is it possible to avoid the counting in the final deterministic check phase of the nondeterministic algorithm without the need of guessing more bits than  $O(\log^2 n)$ ? Or, more in general, without exceeding the upper bound of  $O(\log^2 n)$  nondeterministic guessed bits, is it possible to devise a final deterministic test requiring a formula with strictly less quantifier alternations than those of Condition (3) of Lemma 4.1?

In the quest of finding the exact complexity of the DUAL problem, there is another direction of investigation that could be interesting to be explored.

Flum, Grohe, and Weyer [21] (see also [20]) defined a hierarchy of nondeterministic classes containing those languages that, after guessing  $O(\log^2 n)$  bits, can be checked by an FO formula with a bounded number of quantifier alternations. A different definition of this very same hierarchy can be found also in the paper by Cai and Chen [8]. For notational convenience, let us denote these classes by  $\text{GC}(\log^2 n, \mathcal{S})$ , where  $\mathcal{S}$  is a sequence of logical quantifiers characterizing the quantifiers alternation in the formulas for the check of the languages in the class. Interestingly, there are natural decision problems that are complete for classes in this hierarchy, for example the *tournament dominating set problem*, and the *Vapnik–Chervonenkis dimension problem*.

The former problem is defined as follows: Given a tournament  $G$  (i.e., a directed graph such that for each pair of vertices  $v$  and  $w$  there is either an edge from  $v$  to  $w$ , or an edge from  $w$  to  $v$  (but not both)) and an integer  $k$ , decide whether there is an independent set in  $G$  of size at least  $k$ . It can be shown [8, 20, 21] that this problem is complete for the class  $\text{GC}(\log^2 n, \forall\exists)$ .

The latter problem is defined as follows: Given a hypergraph  $\mathcal{G}$  and an integer  $k$ , decide whether the Vapnik–Chervonenkis dimension of  $\mathcal{G}$  is at least  $k$ . It can be shown [8, 20, 21] that this problem is complete for the class  $\text{GC}(\log^2 n, \forall\exists\forall)$ .

In fact, a new hierarchy of classes characterized by limited nondeterminism could be defined. Indeed, we could extend the definitions given by Cai and Chen [8], and Flum, Grohe, and Weyer [21], to define the classes of languages that, after a nondeterministic guess of  $O(\log^2 n)$  bits, can be checked by a  $\text{FO}(\text{COUNT})$  formula with a bounded number of quantifier alternations.

In particular, for  $\overline{\text{DUAL}}$ , we hypothesize that the formula checking the guess of our nondeterministic algorithm, possibly rearranged and rewritten, is characterized by a quantifiers alternation  $\forall\exists\text{C}$  (where C is the counting quantifier).<sup>10</sup>

Given such a new hierarchy, it would be interesting to verify whether  $\overline{\text{DUAL}}$  belongs to the class  $\text{GC}(\log^2 n, \forall\exists\text{C})$ , and even whether  $\overline{\text{DUAL}}$  is complete for this class.

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<sup>10</sup>Note here that  $\text{FO}(\text{COUNT})$  formulas with bounded quantifiers could be put in relation with the levels of the logarithmic time counting hierarchy defined by Torán [52, 53].

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## A Summary of the definitions, notions, and notations

**Addition of assignments:** See  $\sigma_1 + \sigma_2$ .

**Appealing vertex to exclude:** See vertex appealing to exclude.

**Appealing vertex to include:** See vertex appealing to include.

**Assignment:** Given an instance  $\langle \mathcal{G}, \mathcal{H} \rangle$  of DUAL, an assignment  $\sigma = \langle In, Ex \rangle$  is a pair of subsets of the vertices  $V$  of the hypergraphs such that  $In \cap Ex = \emptyset$ .

**Assignment extension of another:** See  $\sigma_1 \sqsubseteq \sigma_2$ .

**Assignment proper extension of another:** See  $\sigma_1 \sqsubset \sigma_2$ .

**Augmented assignment:** See  $\sigma^+$ .

**Coherence of an assignment with a set:** See  $\sigma \sqsubseteq \sigma_S$ .

**Com( $\sigma$ ):** Given an instance  $\langle \mathcal{G}, \mathcal{H} \rangle$  of DUAL and an assignment  $\sigma = \langle In, Ex \rangle$ ,  $Com(\sigma)$  is the set of the compatible edges of  $\mathcal{H}$ , i.e.,  $Com(\sigma) = \{H \in \mathcal{H} \mid H \cap Ex = \emptyset\}$ .

**Compatible edge:** See  $Com(\sigma)$ .

**Complement set:** See  $\overline{S}$ .

**Consistent set of labels:** See  $\sigma(\Sigma)$ .

**Cov( $\sigma$ ):** Given an instance  $\langle \mathcal{G}, \mathcal{H} \rangle$  of DUAL and an assignment  $\sigma = \langle In, Ex \rangle$ ,  $Cov(\sigma)$  is the set of the edges of  $\mathcal{H}$  entirely covered by  $\sigma$ , i.e.,  $Cov(\sigma) = \{H \in \mathcal{H} \mid H \subseteq In\}$ .

**Covering assignment:** Given an instance  $\langle \mathcal{G}, \mathcal{H} \rangle$  of DUAL, an assignment  $\sigma = \langle In, Ex \rangle$  is covering if  $In$  or  $Ex$  are covering.

**Covering  $Ex$ :** Given an instance  $\langle \mathcal{G}, \mathcal{H} \rangle$  of DUAL and an assignment  $\sigma = \langle In, Ex \rangle$ ,  $Ex$  is covering if there exists an edge  $G \in \mathcal{G}$  such that  $G \subseteq Ex$ .

**Covering  $In$ :** Given an instance  $\langle \mathcal{G}, \mathcal{H} \rangle$  of DUAL and an assignment  $\sigma = \langle In, Ex \rangle$ ,  $In$  is covering if there exists an edge  $H \in \mathcal{H}$  such that  $H \subseteq In$ .

**Critical vertex:** Given a hypergraph  $\mathcal{G} = \langle V, E \rangle$  and a set of vertices  $T \subseteq V$ , a vertex  $v \in V$  is critical in  $T$  w.r.t.  $\mathcal{G}$  if there exists an edge  $G \in \mathcal{G}$  such that  $T \cap G = \{v\}$ .

**Double witness (of the existence of a new transversal):** Given an instance  $\langle \mathcal{G}, \mathcal{H} \rangle$  of DUAL, an assignment  $\sigma = \langle In, Ex \rangle$  is a double witness of the existence of a new transversal of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$  if  $In$  is a new transversal of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$  and  $Ex$  is a new transversal of  $\mathcal{H}$  w.r.t.  $\mathcal{G}$ .

**DSPACE[ $f(n)$ ]** : The class of the languages that can be decided by deterministic Turing machines using  $O(f(n))$  work-tape cells.

**DUAL:** Denotes the decision problem of hypergraph duality, that is, given a pair of hypergraphs  $\langle \mathcal{G}, \mathcal{H} \rangle$  decide whether  $\mathcal{H} = tr(\mathcal{G})$ .

$\overline{\text{DUAL}}$ : Denotes the decision problem complement to DUAL.

**Empty assignment:** See  $\sigma_\varepsilon$ .

**Empty hypergraph:** A hypergraph  $\mathcal{G}$  is an empty hypergraph if  $\mathcal{G} = \langle V, \emptyset \rangle$ . In this paper we assume that hypergraphs are not empty unless it is differently stated.

**Empty-edge hypergraph:** A hypergraph  $\mathcal{G}$  is an empty-edge hypergraph if  $\mathcal{G} = \langle V, \{\emptyset\} \rangle$ . In this paper we assume that hypergraphs are not empty-edge unless it is differently stated.

**End-node of a path:** See  $\mathcal{N}(\Pi)$ .

**Entirely covered edge:** See  $Cov(\sigma)$ .

**Entirely missed edge:** See  $Mis(\sigma)$ .

**Ex( $\Sigma$ ):** See  $\sigma(\Sigma)$ .

**Ex<sub>p</sub>:** See  $\sigma_p$ .

**Excluded vertex (of an assignment):** Given an assignment  $\sigma = \langle In, Ex \rangle$ ,  $v$  is an excluded vertex of  $\sigma$  if  $v \in Ex$ .

**Excluding a vertex as a critical vertex into an assignment:** See Extension type (iv).

**Excluding a vertex into an assignment:** See Extension type (iii).

**Extension type (i):** Given an instance  $\langle \mathcal{G}, \mathcal{H} \rangle$  of DUAL and an assignment  $\sigma = \langle In, Ex \rangle$ , in the quest of finding a new transversal of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$ , a vertex  $v$  is included into the attempted new transversal, i.e.,  $\sigma' = \sigma + \langle \{v\}, \emptyset \rangle$ .

**Extension type (ii):** Given an instance  $\langle \mathcal{G}, \mathcal{H} \rangle$  of DUAL and an assignment  $\sigma = \langle In, Ex \rangle$ , in the quest of finding a new transversal of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$ , a vertex  $v$  along with an edge  $G \in \mathcal{G}(\sigma)$  (such that  $v \in G$ ) witnessing  $v$ 's criticality is included as a critical vertex into the attempted new transversal, i.e.,  $\sigma' = \sigma + \langle \{v\}, G \setminus \{v\} \rangle$ .

**Extension type (iii):** Given an instance  $\langle \mathcal{G}, \mathcal{H} \rangle$  of DUAL and an assignment  $\sigma = \langle In, Ex \rangle$ , in the quest of finding a new transversal of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$ , a vertex  $v$  is excluded in the attempted new transversal (or, equivalently,  $v$  is included into the attempted new transversal of  $\mathcal{H}$  w.r.t.  $\mathcal{G}$ ), i.e.,  $\sigma' = \sigma + \langle \emptyset, \{v\} \rangle$ .

**Extension type (iv):** Given an instance  $\langle \mathcal{G}, \mathcal{H} \rangle$  of DUAL and an assignment  $\sigma = \langle In, Ex \rangle$ , in the quest of finding a new transversal of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$ , a vertex  $v$  along with an edge  $H \in \mathcal{H}(\sigma)$  (such that  $v \in H$ ) witnessing  $v$ 's criticality is included as a critical vertex into the attempted new transversal of  $\mathcal{H}$  w.r.t.  $\mathcal{G}$ , i.e.,  $\sigma' = \sigma + \langle H \setminus \{v\}, \{v\} \rangle$ .

**FO:** First order logic.

**FO(COUNT):** First order logic augmented by the counting quantifier.

**FOM:** First order logic augmented by the majority quantifier.

**Free vertex (of an assignment):** Given an assignment  $\sigma = \langle In, Ex \rangle$ ,  $v$  is a free vertex of  $\sigma$  if  $v \notin In$  and  $v \notin Ex$ .

**Freq( $\sigma$ ):** Denotes the set of all the frequent vertices of  $\sigma$ .

**Frequent vertex (of an assignment):** Given an instance  $\langle \mathcal{G}, \mathcal{H} \rangle$  of DUAL and an assignment  $\sigma$ , a free vertex of  $\sigma$  is frequent in  $\sigma$  if it belongs to at least half of the edges of  $Com(\sigma)$ .

**GC( $f(n)$ ,  $\mathcal{C}$ ):** Denotes the guess-and-check complexity class of all problems that after a nondeterministic guess of  $O(f(n))$  bits can be decided (checked) in complexity class  $\mathcal{C}$ .

**$|\mathcal{G}|$ :** Denotes the number of edges of the hypergraph  $\mathcal{G}$ .

**$\|\mathcal{G}\|$ :** Denotes the size of the hypergraph  $\mathcal{G}$ , that is the space (in terms of the number of bits) required to represent  $\mathcal{G}$ .

**$\mathcal{G} \subseteq \mathcal{H}$ :** Given two hypergraphs  $\mathcal{G} = \langle V, E \rangle$  and  $\mathcal{H} = \langle W, F \rangle$ ,  $\mathcal{G} \subseteq \mathcal{H}$  denotes  $V \subseteq W$  and  $E \subseteq F$ .

**$\mathcal{G}(\sigma)$ :** See  $\mathcal{I}_\sigma$ .

**$\mathcal{G}_S$ :** Given a hypergraph  $\mathcal{G}$ ,  $\mathcal{G}_S = \langle S, \{G \in \mathcal{G} \mid G \subseteq S\} \rangle$ . Observe that if  $\mathcal{G}$  is simple, then  $\mathcal{G}_S$  is also simple.

**$\mathcal{G}^S$ :** Given a hypergraph  $\mathcal{G}$ ,  $\mathcal{G}_S = \langle S, \min(\{G \cap S \mid G \in \mathcal{G}\}) \rangle$ . Observe that  $\mathcal{G}^S$  is always simple by definition.

**$\mathcal{H}(\sigma)$ :** See  $\mathcal{I}_\sigma$ .

**Hypergraph:** A hypergraph  $\mathcal{G} = \langle V, E \rangle$  is a pair where  $V$  is its non-empty set of vertex and  $E \subseteq 2^V$  is its non-empty set of edges. We often identify a hypergraph  $\mathcal{G} = \langle V, E \rangle$  with its edge-set and vice-versa. That is, by writing  $G \in \mathcal{G}$  we mean  $G \in E$ , and by writing  $\mathcal{G} = E$  we mean that  $\mathcal{G} = \langle \bigcup_{G \in E} G, E \rangle$ . If it is not stated otherwise, in this paper it is presupposed that hypergraphs have the same set of vertices, which is denoted by  $V$ , and that each vertex of a hypergraph belongs to at least one of its edges.

**$\mathcal{I}_\sigma$ :** Given an instance  $\mathcal{I} = \langle \mathcal{G}, \mathcal{H} \rangle$  of DUAL and an assignment  $\sigma = \langle In, Ex \rangle$ , the reduced instance of  $\mathcal{I}$  on  $\sigma$  is  $\mathcal{I}_\sigma = \langle \mathcal{G}(\sigma), \mathcal{H}(\sigma) \rangle = \langle (\mathcal{G}_{V \setminus In})^{V \setminus (In \cup Ex)}, (\mathcal{H}_{V \setminus Ex})^{V \setminus (In \cup Ex)} \rangle$ .

**$In(\Sigma)$ :** See  $\sigma(\Sigma)$ .

**$In_p$ :** See  $\sigma_p$ .

**Included vertex (of an assignment):** Given an assignment  $\sigma = \langle In, Ex \rangle$ ,  $v$  is an included vertex of  $\sigma$  if  $v \in In$ .

**Including a vertex as a critical vertex into an assignment:** See Extension type (ii).

**Including a vertex into an assignment:** See Extension type (i).

**Independent set (of a hypergraph):** Given a hypergraph  $\mathcal{G}$ , a set of vertices  $S \subseteq V$  is an independent set of  $\mathcal{G}$  if, for every edge  $G \in \mathcal{G}$ ,  $G \not\subseteq S$ .

**$Infreq(\sigma)$ :** Denotes the set of all the infrequent vertices of  $\sigma$ .

**Infrequent vertex (of an assignment):** Given an instance  $\langle \mathcal{G}, \mathcal{H} \rangle$  of DUAL and an assignment  $\sigma$ , a free vertex of  $\sigma$  is infrequent in  $\sigma$  if it belongs to less than half of the edges of  $Com(\sigma)$ .

**Intersection property:** Two hypergraphs  $\mathcal{G}$  and  $\mathcal{H}$  satisfy the intersection property if every edge of  $\mathcal{G}$  has a non-empty intersection with every edge of  $\mathcal{H}$ , and vice-versa.

**$\mathcal{L}(p)$ :** Given an instance  $\langle \mathcal{G}, \mathcal{H} \rangle$  of DUAL and a node  $p$  of the tree  $\mathcal{T}(\mathcal{G}, \mathcal{H})$ ,  $\mathcal{L}(p)$  denotes the set of the labels of the edges leaving  $p$ .

**$\ell(a)$ :** Given an instance  $\langle \mathcal{G}, \mathcal{H} \rangle$  of DUAL and an edge  $a$  of the tree  $\mathcal{T}(\mathcal{G}, \mathcal{H})$ ,  $\ell(a)$  denotes the label of the edge  $a$ .

**LOGSPACE:** Denotes the class of languages that can be decided by deterministic Turing machines within logarithmic space bound.

**$m$ :** Given an instance  $\langle \mathcal{G}, \mathcal{H} \rangle$  of DUAL,  $m = |\mathcal{G}| + |\mathcal{H}|$ .

**$\min(\mathcal{G})$ :** Given a hypergraph  $\mathcal{G}$ ,  $\min(\mathcal{G})$  denotes the set of inclusion minimal edges of  $\mathcal{G}$ .

**Minimal transversal (of a hypergraph):** Given a hypergraph  $\mathcal{G}$ , a set of vertices  $T \subseteq V$  is a minimal transversal of  $\mathcal{G}$  if, for every proper subset  $T'$  of  $T$ ,  $T'$  is not a transversal of  $\mathcal{G}$ .

**$Mis(\sigma)$ :** Given an instance  $\langle \mathcal{G}, \mathcal{H} \rangle$  of DUAL and an assignment  $\sigma = \langle In, Ex \rangle$ ,  $Mis(\sigma)$  is the set of the edges of  $\mathcal{G}$  entirely missed by  $\sigma$ , i.e.,  $Mis(\sigma) = \{G \in \mathcal{G} \mid G \subseteq Ex\}$ .

**$N$ :** Given an instance  $\langle \mathcal{G}, \mathcal{H} \rangle$  of DUAL,  $N = \|\mathcal{G}\| + \|\mathcal{H}\|$  denotes the size to represent the instance.



**$\mathcal{N}(\Pi)$ :** Given an instance  $\langle \mathcal{G}, \mathcal{H} \rangle$  of DUAL,  $\mathcal{N}(\Pi)$  denotes the node of the tree  $\mathcal{T}(\mathcal{G}, \mathcal{H})$  that is reached following successively the labels in the path  $\Pi$ .

**New transversal (of a hypergraph w.r.t. another):** Given two hypergraphs  $\mathcal{G}$  and  $\mathcal{H}$ , a set of vertices  $T \subseteq V$  is a new transversal of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$  if  $T$  is a transversal of  $\mathcal{G}$  and  $T$  is an independent set of  $\mathcal{H}$  (i.e.,  $T$  is a transversal of  $\mathcal{G}$  missing in  $\mathcal{H}$ ).

**Path (in  $\mathcal{T}(\mathcal{G}, \mathcal{H})$ ):** Given an instance  $\langle \mathcal{G}, \mathcal{H} \rangle$  of DUAL, a path  $\Pi = \{\ell_1, \dots, \ell_k\}$  is a sequence of labels in the tree  $\mathcal{T}(\mathcal{G}, \mathcal{H})$  describing the path from the root of  $\mathcal{T}(\mathcal{G}, \mathcal{H})$  to the node reached by following the edges labeled in turn  $\ell_1, \ell_2, \dots, \ell_k$ .

**Precursor (of a new transversal):** Given an instance  $\langle \mathcal{G}, \mathcal{H} \rangle$  of DUAL, if  $T$  is a new transversal of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$ , an assignment  $\sigma$  is a precursor of  $T$  if  $\sigma \sqsubset T$  and  $\sigma$  is not a witness.

**PTIME:** Denotes the class of languages that can be decided by deterministic Turing machines within a polynomial time bound.

**Ratio of the edges containing a vertex:** See  $\varepsilon_v^{\mathcal{G}}$ .

**Reduced instance of DUAL:** See  $\mathcal{I}_\sigma$ .

**Reversed assignment:** See  $\bar{\sigma}$ .

**$\bar{S}$ :** Given a hypergraph  $\mathcal{G} = \langle V, E \rangle$ , for a set of vertices  $S \subseteq V$ ,  $\bar{S}$  denotes the complement of  $S$  in  $V$ , i.e.,  $\bar{S} = V \setminus S$ .

**$\mathcal{S}^{\log}(\mathcal{G}, \mathcal{H})$ :** Given an instance  $\langle \mathcal{G}, \mathcal{H} \rangle$ ,  $\mathcal{S}^{\log}(\mathcal{G}, \mathcal{H})$  denotes the set of all the sets containing logarithmic-many labels of edges of the tree  $\mathcal{T}(\mathcal{G}, \mathcal{H})$ . More precisely,  $\mathcal{S}^{\log}(\mathcal{G}, \mathcal{H}) = \{\Sigma \mid \Sigma = \{\ell_1, \dots, \ell_k\} \text{ is a set of labels such that } (0 \leq k \leq \lfloor \log |\mathcal{H}| \rfloor + 1) \wedge (\ell_i \in \mathcal{L}(r), \forall 1 \leq i \leq k)\}$ , where  $r$  is the root node of  $\mathcal{T}(\mathcal{G}, \mathcal{H})$ .

**Saturated precursor (of a new minimal transversal):** Given an instance  $\langle \mathcal{G}, \mathcal{H} \rangle$  of DUAL, if  $T$  is a new minimal transversal of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$ , an assignment  $\sigma$  is a saturated precursor of  $T$  if  $\sigma$  is a precursor of  $T$  and no free vertex of  $\sigma$  is appealing to exclude or include for  $\sigma$  w.r.t.  $T$ .

**$Sep(\sigma)$ :** Given an instance  $\langle \mathcal{G}, \mathcal{H} \rangle$  of DUAL and an assignment  $\sigma = \langle In, Ex \rangle$ ,  $Sep(\sigma)$  is the set of the separated edges of  $\mathcal{G}$ , i.e.,  $Sep(\sigma) = \{G \in \mathcal{G} \mid G \cap In = \emptyset\}$ .

**Separated edge:** See  $Sep(\sigma)$ .

**Set of labels (of  $\mathcal{T}(\mathcal{G}, \mathcal{H})$ ):** Given an instance  $\langle \mathcal{G}, \mathcal{H} \rangle$  of DUAL, a set of labels of  $\mathcal{T}(\mathcal{G}, \mathcal{H})$  is a set of labels appearing on the edges of  $\mathcal{T}(\mathcal{G}, \mathcal{H})$ . Please note the difference with a path in  $\mathcal{T}(\mathcal{G}, \mathcal{H})$ : a path has to be a valid sequence of labels actually denoting a path in  $\mathcal{T}(\mathcal{G}, \mathcal{H})$ , instead a set of labels does not have to satisfy this condition (it is just a set); moreover the path is a sequence (the order matters), while for a set of labels the order does not matter (again, it is just a set).

**Simple hypergraph:** A hypergraph  $\mathcal{G}$  is simple whenever, for every pair of distinct edges  $G_1, G_2 \in \mathcal{G}$ ,  $G_1 \not\subseteq G_2$ .

**$\mathcal{T}(\mathcal{G}, \mathcal{H})$ :** Given an instance  $\langle \mathcal{G}, \mathcal{H} \rangle$  of DUAL,  $\mathcal{T}(\mathcal{G}, \mathcal{H})$  denotes the assignment/decomposition tree of the instance  $\langle \mathcal{G}, \mathcal{H} \rangle$  in which all the possible decompositions according to the extension types (ii) and (iii) are considered. The leaves of  $\mathcal{T}(\mathcal{G}, \mathcal{H})$  are those sub-instances associated with covering assignments or assignments without free vertices. For more details please see the definition given in Section 3.2 (at page 7).

**$TC^0$ :** The complexity class of languages that can be decided by circuits of polynomial size and constant depth characterized by AND, OR, NOT, and thresholds gates of unbounded fan-in. It is well known that log-time-uniform  $TC^0$  is equivalent to  $FO(COUNT)$ .

**$tr(\mathcal{G})$ :** See Transversal/Dual hypergraph.

**Transversal (of a hypergraph):** Given a hypergraph  $\mathcal{G} = \langle V, E \rangle$ , a set of vertices  $T \subseteq V$  is a transversal of  $\mathcal{G}$  if, for every edge  $G \in \mathcal{G}$ ,  $T \cap G \neq \emptyset$ .

**Transversal/Dual hypergraph:** Given a hypergraph  $\mathcal{G}$ , hypergraph  $\mathcal{H}$  is the transversal, or the dual, hypergraph of  $\mathcal{G}$  if the edges of  $\mathcal{H}$  are all and only the minimal transversals of  $\mathcal{G}$ , and we denote this by  $\mathcal{H} = tr(\mathcal{G})$ . In this case  $\mathcal{G}$  and  $\mathcal{H}$  are also said to be dual. It is well known that  $\mathcal{H} = tr(\mathcal{G})$  iff  $\mathcal{G} = tr(\mathcal{H})$ .

**Trivially dual hypergraphs:** Two hypergraphs are trivially dual if one of them is the empty hypergraph and the other is the empty-edge hypergraph.

**Vertex appealing to exclude (for an assignment  $\sigma$ ) w.r.t.  $T$ :** Given an instance  $\langle \mathcal{G}, \mathcal{H} \rangle$  of DUAL, if  $T$  is a new minimal transversal of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$  and  $\sigma$  is a precursor of  $T$ , a free vertex  $v$  of  $\sigma$  is appealing to exclude for  $\sigma$  if  $v \in \text{Freq}(\sigma)$ , and  $v \notin T$ .

**Vertex appealing to include (for an assignment  $\sigma$ ) w.r.t.  $T$ :** Given an instance  $\langle \mathcal{G}, \mathcal{H} \rangle$  of DUAL, if  $T$  is a new minimal transversal of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$  and  $\sigma$  is a precursor of  $T$ , a free vertex  $v$  of  $\sigma$  is appealing to include for  $\sigma$  if  $v \in \text{Infreq}(\sigma)$ , and  $v \in T$ .

**Witness (of the existence of a new transversal):** Given an instance  $\langle \mathcal{G}, \mathcal{H} \rangle$  of DUAL, an assignment  $\sigma = \langle \text{In}, \text{Ex} \rangle$  is a witness of the existence of a new transversal of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$  if  $\text{In}$  is a new transversal of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$  or  $\text{Ex}$  is a new transversal of  $\mathcal{H}$  w.r.t.  $\mathcal{G}$ .

**$-v$ :** Given an instance  $\langle \mathcal{G}, \mathcal{H} \rangle$  of DUAL and a vertex  $v \in V$ ,  $-v$  is the label of the edges of the tree  $\mathcal{T}(\mathcal{G}, \mathcal{H})$  associated with the application of the extension type (iii) in which vertex  $v$  is excluded.

**$(v, G)$ :** Given an instance  $\langle \mathcal{G}, \mathcal{H} \rangle$  of DUAL, a vertex  $v \in V$ , and an edge  $G \in \mathcal{G}$  such that  $v \in G$ ,  $(v, G)$  is the label of the edges of the tree  $\mathcal{T}(\mathcal{G}, \mathcal{H})$  associated with the application of the extension type (ii) in which vertex  $v$  is included as a critical vertex along with edge  $G$  witnessing  $v$ 's criticality.

**$\varepsilon_v^{\mathcal{G}}$ :** Given a hypergraph  $\mathcal{G} = \langle V, E \rangle$  and a vertex  $v \in V$ ,  $\varepsilon_v^{\mathcal{G}}$  is the ratio of the edges of  $\mathcal{G}$  containing  $v$ , i.e.,  $\varepsilon_v^{\mathcal{G}} = \frac{|\{G \in E \mid v \in G\}|}{|E|}$ . Sometimes in the paper this notation is used also when the superscript “ $\mathcal{G}$ ” is not, strictly speaking, a hypergraph but instead it is just a set of edges.

**$\bar{\sigma}$ :** Given an assignment  $\sigma = \langle \text{In}, \text{Ex} \rangle$ ,  $\bar{\sigma} = \langle \text{Ex}, \text{In} \rangle$ .

**$\sigma^+$ :** Given an instance  $\langle \mathcal{G}, \mathcal{H} \rangle$  of DUAL,  $\sigma^+$  is the augmented assignment of  $\sigma = \langle \text{In}, \text{Ex} \rangle$  and it is defined as  $\sigma^+ = \langle \text{In} \cup \text{Freq}(\sigma), \text{Ex} \cup \text{Infreq}(\sigma) \rangle$ .

**$\sigma(\Sigma)$ :** Given an instance  $\langle \mathcal{G}, \mathcal{H} \rangle$  of DUAL and set  $\Sigma$  of labels of the tree  $\mathcal{T}(\mathcal{G}, \mathcal{H})$ ,  $\sigma(\Sigma) = \langle \text{In}(\Sigma), \text{Ex}(\Sigma) \rangle$  is a pair of sets of vertices of the hypergraphs in which  $\text{In}(\Sigma) = \bigcup_{(v, G) \in \Sigma} \{v\}$  and  $\text{Ex}(\Sigma) = (\bigcup_{-v \in \Sigma} \{v\}) \cup (\bigcup_{(v, G) \in \Sigma} (G \setminus \{v\}))$ . If  $\text{In}(\Sigma) \cap \text{Ex}(\Sigma) = \emptyset$  then the set  $\Sigma$  is said to be consistent. Regardless of whether  $\Sigma$  is consistent, in the text  $\sigma(\Sigma)$  is treated as an assignment and the notations and definitions given for assignments are extended to  $\sigma(\Sigma)$ .

**$\sigma(\Pi)$ :** If  $\Pi$  is a path in the tree  $\mathcal{T}(\mathcal{G}, \mathcal{H})$ , see  $\sigma_{\mathcal{N}(\Pi)}$ . If  $\Pi$  is a set of labels, see  $\sigma(\Sigma)$ .

**$\sigma \sqsubseteq \sigma_S$ :** See  $\sigma \sqsubseteq \sigma_S$ .

**$\sigma_{\mathcal{N}(\Pi)}$ :** Given an instance  $\langle \mathcal{G}, \mathcal{H} \rangle$  of DUAL,  $\sigma_{\mathcal{N}(\Pi)}$  denotes the assignment of the node  $\mathcal{N}(\Pi)$  identified by the path  $\Pi$  in the tree  $\mathcal{T}(\mathcal{G}, \mathcal{H})$ .  $\sigma_{\mathcal{N}(\Pi)}$  is also denoted by  $\sigma(\Pi)$ . By the definitions of the extension types,  $\sigma_{\mathcal{N}(\Pi)} = \sigma(\Pi) = \langle \bigcup_{(v, G) \in \Pi} \{v\}, (\bigcup_{-v \in \Pi} \{v\}) \cup (\bigcup_{(v, G) \in \Pi} (G \setminus \{v\})) \rangle$ .

**$\sigma_p$ :** Given an instance  $\langle \mathcal{G}, \mathcal{H} \rangle$  of DUAL,  $\sigma_p = \langle \text{In}_p, \text{Ex}_p \rangle$  is the assignment associated with the node  $p$  of the tree  $\mathcal{T}(\mathcal{G}, \mathcal{H})$ .

**$\sigma \sqsubseteq \sigma_S$ :** Given an assignment  $\sigma = \langle \text{In}, \text{Ex} \rangle$  and a set of vertices  $S$ ,  $\sigma \sqsubseteq \sigma_S$  denotes that  $\text{In} \subseteq S$  and  $\text{Ex} \subseteq \bar{S}$  (or, equivalently,  $\text{Ex} \cap S = \emptyset$ ). In this case  $\sigma$  is said to be coherent with  $S$ , and vice-versa. The condition  $\sigma \sqsubseteq \sigma_S$  is also denoted by  $\sigma \sqsubseteq S$ .

**$\sigma_1 + \sigma_2$ :** Given two assignments  $\sigma_1 = \langle \text{In}_1, \text{Ex}_1 \rangle$  and  $\sigma_2 = \langle \text{In}_2, \text{Ex}_2 \rangle$ , if  $\text{In}_1 \cap \text{Ex}_2 = \emptyset$  and  $\text{Ex}_1 \cap \text{In}_2 = \emptyset$ , then  $\sigma_1 + \sigma_2$  is the assignment  $\langle \text{In}_1 \cup \text{In}_2, \text{Ex}_1 \cup \text{Ex}_2 \rangle$ .

**$\sigma_1 \sqsubset \sigma_2$ :** Given two assignments  $\sigma_1 = \langle \text{In}_1, \text{Ex}_1 \rangle$  and  $\sigma_2 = \langle \text{In}_2, \text{Ex}_2 \rangle$ ,  $\sigma_1 \sqsubset \sigma_2$  denotes that  $\sigma_1 \sqsubseteq \sigma_2$  and  $\text{In}_1 \subset \text{In}_2$  or  $\text{Ex}_1 \subset \text{Ex}_2$ . In this case  $\sigma_2$  is said to be a proper extension of  $\sigma_1$ .

**$\sigma_1 \sqsubseteq \sigma_2$ :** Given two assignments  $\sigma_1 = \langle \text{In}_1, \text{Ex}_1 \rangle$  and  $\sigma_2 = \langle \text{In}_2, \text{Ex}_2 \rangle$ ,  $\sigma_1 \sqsubseteq \sigma_2$  denotes that  $\text{In}_1 \subseteq \text{In}_2$  and  $\text{Ex}_1 \subseteq \text{Ex}_2$ . In this case  $\sigma_2$  is said to be an extension of  $\sigma_1$ .

**$\sigma_S$ :** Given a set of vertices  $S$ ,  $\sigma_S = \langle S, \bar{S} \rangle$ .

**$\sigma_\varepsilon$ :**  $\sigma_\varepsilon = \langle \emptyset, \emptyset \rangle$ .

## B It's a matter of perspectives...

In the literature, the hypergraph transversal problem was faced essentially from two points of view: that of Boolean functions dualization, and that of hypergraphs themselves. In order to ease the (non-expert) reader's task to place this work in the literature's landscape we illustrate here their connections.

Consider the *Boolean domain*  $\{0,1\}$ . A *Boolean vector* is an element of the  $n$ -dimensional Boolean space  $\{0,1\}^n$ . If  $x$  is a Boolean vector,  $x_i$  denotes the  $i$ -th component of  $x$ . If  $x$  and  $y$  are two vectors belonging to the same  $n$ -dimensional Boolean space (and hence having the same number of components), by  $x \leq y$  we denote the fact that  $x_i \leq y_i$  for all  $1 \leq i \leq n$ .

An  $n$ -ary *Boolean function*  $f$  is a mapping  $f: \{0,1\}^n \mapsto \{0,1\}$  from the  $n$ -dimensional Boolean space to a Boolean value. Given two functions  $f$  and  $g$  defined on the same  $n$ -dimensional Boolean space, with  $f \leq g$  we mean that  $f(x) \leq g(x)$  for all the Boolean vectors  $x \in \{0,1\}^n$  (i.e., for all the vectors of the domain). A function  $f$  is said to be *monotone* (or *positive*), if, for any two vectors  $x, y \in \{0,1\}^n$ ,  $x \leq y$  implies  $f(x) \leq f(y)$ .

A way to represent an  $n$ -ary Boolean function is through a *Boolean formula* in  $n$  variables  $x_1, \dots, x_n$ . Boolean variables  $x_1, \dots, x_n$  and their complements  $\neg x_1, \dots, \neg x_n$  are called *literals*. A Boolean formula is a formula consisting of Boolean constants 0 and 1 (which we associate with **false** and **true**, respectively), literals, logical connectives ' $\wedge$ ' (logical *and*, or *conjunction*), and ' $\vee$ ' (logical *or*, or *disjunction*), and the parentheses symbols '(' and ')'. Parentheses give priority to the evaluation of the subformula between them enclosed, altering, in this way, the standard precedence of the conjunction over the disjunction.

A *clause* and a *term* are a disjunction and a conjunction of literals, respectively. A clause  $c$  is an *implicate* of the function  $f$  if  $f \leq c$ , while a term  $t$  is an *implicant* of  $f$  if  $t \leq f$ . A clause  $c$  is a *prime implicate* of  $f$ , and a term  $t$  is a *prime implicant* of  $f$ , if they are minimal. Here "minimal" means that they are no longer an implicate of  $f$  and an implicant of  $f$ , respectively, if a literal is removed from them. Implicates and implicants are said to be *monotone* if they consist only of positive literals.

A Boolean formula is said to be in *conjunctive normal form* (or, *CNF*) if it is a conjunction of clauses, while it is said to be in *disjunctive normal form* (or, *DNF*) if it is a disjunction of terms. A CNF, or a DNF, formula is said to be *prime* or *monotone* if all its clauses, or terms, are prime or monotone, respectively.

It is well known that, a function  $f$  is monotone if and only if it can be represented through a monotone CNF, or a monotone DNF, formula. These representations are moreover unique if only prime implicates and implicants are considered. Given a monotone Boolean function  $f$ , we denote by  $CNF(f)$  and  $DNF(f)$  its unique prime CNF and DNF representations, respectively.

If  $f$  is a Boolean function, we denote by  $f^d$  its *dual* that is defined as  $f^d(x) = \neg f(\neg x)$ , where  $\neg f$  and  $\neg x$  are the complements of  $f$  and  $x$  (each components of  $x$  is complemented), respectively. By definition, a function  $f$  and its dual  $f^d$  are such that  $(f^d)^d = f$ . If  $f$  is a monotone Boolean function, then also  $f^d$  is monotone. Let  $f$  be a Boolean function, and  $\gamma$  be a Boolean formula representing  $f$ . From De Morgan's law, it is possible to easily compute a Boolean formula representing  $f^d$  by simply exchanging the logical connectives  $\wedge$  and  $\vee$ , and the constants 0 and 1, of  $\gamma$ . We denote by  $\gamma^\uparrow$  the Boolean formula obtained from  $\gamma$  by exchanging its logical connectives and constants. Observe that, if  $f$  is a monotone Boolean function, and  $\varphi = CNF(f)$ , then  $\varphi^\uparrow = DNF(f^d)$  (and, obviously, if  $\psi = DNF(f)$  then  $\psi^\uparrow = CNF(f^d)$ ). This means that it is trivial to compute from  $CNF(f)$  ( $DNF(f)$ , resp.) the formula  $DNF(f^d)$  ( $CNF(f^d)$ , resp.). In fact, it is a more involved task to compute from  $CNF(f)$  the formula  $CNF(f^d)$  (and from  $DNF(f)$  the formula  $DNF(f^d)$ ; a task that has the very same complexity).

With the aim of studying the computational complexity of the dual function computation problem, its decision variant was introduced in the literature. This decision task consists in determining whether two given monotone prime CNF, or DNF, formulas represent dual functions.

To go back to the hypergraph transversal problem, one of the forms in which the transversal problem was approached in the literature is through the *dualization* or the *duality* (that is, the decision problem) perspective. Among the vast literature published so far on the topic, we find that computing the formula  $CNF(f^d)$  from  $CNF(f)$  (or its decision flavour, that is, given two CNF formulas deciding whether they are dual) was studied, for example, by Eiter et al. [16, 17]; and that computing the formula  $DNF(f^d)$  from  $DNF(f)$  (or its decision flavour, that is, given two DNF formulas deciding whether they are dual) was studied, for example, by Fredman and Khachiyan [22], and Kavvadias and Stavropoulos [37, 38].

Since the dual of a monotone Boolean function  $f$  is itself monotone, if  $\psi = CNF(f^d)$  then  $\psi^\uparrow = DNF(f)$ . This implies that it is trivial to derive the DNF formula of a function  $f$  if it has been previously computed the CNF formula of the dual function  $f^d$  (and vice-versa). For this reason, sometimes the problem of dualization was approached as the problem of computing the formula  $DNF(f)$  when the formula  $CNF(f)$  is given in input (or vice-versa). Also in this case there is a decision counterpart of this problem. Given two monotone prime Boolean formulas, one in CNF and the other in DNF, deciding whether they represent equivalent functions. Dealing with the transversal problem from this point of view was an approach adopted, for example, by Elbassioni [18], and Boros et al. [7].

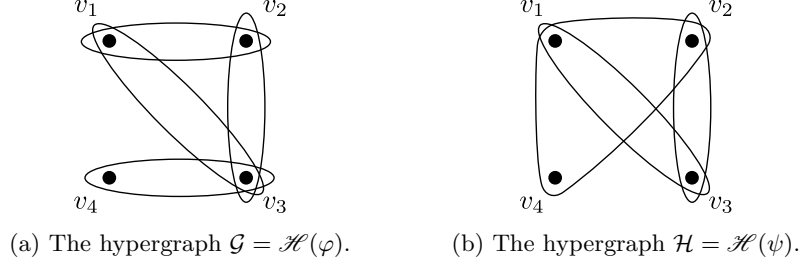


Figure 4: The hypergraphs obtained from  $\varphi = (x_1 \vee x_2) \wedge (x_2 \vee x_3) \wedge (x_1 \vee x_3) \wedge (x_3 \vee x_4)$  and  $\psi = (x_1 \wedge x_2 \wedge x_4) \vee (x_1 \wedge x_3) \vee (x_2 \wedge x_3)$ . Since  $\varphi = \text{CNF}(f)$  and  $\psi = \text{DNF}(f)$ , for the very same monotone Boolean function  $f$ ,  $\mathcal{H}(\psi) = \text{tr}(\mathcal{H}(\varphi))$  (and vice-versa).

Another perspective is that of self-duality, consisting in deciding whether a Boolean function  $f$  is such that  $f^d = f$ . This problem is known to be characterized by the very same complexity of deciding the duality of two Boolean formulas [13]. In the literature, the works of Gaur [24], and Gaur and Krishnamurti [25], approached the transversal problem from this point of view (self-duality of DNF prime monotone Boolean formulas).

Apart from the approaches mentioned above, there are also works in which the authors tackled “directly” the transversal problem dealing explicitly with hypergraphs, vertices, hyperedges, transversal sets and related concepts [4, 12, 13, 19, 28]. Some of these works at first introduce the problem from a Boolean formulas perspective before shifting to an approach more focused on hypergraphs.

Since all these problems are strictly connected each other, sometimes in the literature the transversal hypergraph is called the dual hypergraph. By extension, often the task of computing the transversal/dual hypergraph of a given one is referred to as the dualization of a hypergraph, while the task of deciding whether two given hypergraphs are each the transversal/dual hypergraph of the other is referred to as the task of checking the duality between two hypergraphs.

In this paper we approach the hypergraph transversal problem in a explicit way by dealing with hypergraph “entities”, as vertices and hyperedges. In Section 3 we will refer to vertices included in and excluded from the ongoing building (possibly new) transversal of  $\mathcal{G}$ . We want to relate this notion of including/excluding vertices with the other approaches used in the literature to make clearer the similarities and the differences of the algorithm here presented with the other proposed.

Let  $\varphi$  be a prime monotone CNF formula. Let us denote by  $\mathcal{H}(\varphi)$  the hypergraph obtained from  $\varphi$ , consisting of a vertex  $v_i$  for each variable  $x_i$  of  $\varphi$ , and a hyperedge  $H_c$  for each clause  $c$  of  $\varphi$ . Hyperedge  $H_c$  contains exactly the vertices associated with the literals of the clause  $c$  (remember that all the literals of  $\varphi$  are positive since the formula is assumed to be monotone). For example, let  $f$  be the 4-ary monotone Boolean function such that  $f(x) = 1$  if and only if  $x \in \{(1, 1, 0, 1), (0, 1, 1, 0), (1, 0, 1, 0)\}$ . From this information it is easy to obtain the formula  $\text{DNF}(f)$ , but, for the sake of the presentation, let us first show the formula  $\text{CNF}(f)$ . We ask the reader at first to “trust” that it is correct, we will prove it shortly. Let

$$\varphi = \text{CNF}(f) = \underbrace{(x_1 \vee x_2)}_{c_1} \wedge \underbrace{(x_2 \vee x_3)}_{c_2} \wedge \underbrace{(x_1 \vee x_3)}_{c_3} \wedge \underbrace{(x_3 \vee x_4)}_{c_4},$$

the hypergraph  $\mathcal{G} = \mathcal{H}(\varphi)$  is depicted in Figure 4a. Note that for any prime monotone CNF formula  $\phi$ , the hypergraph  $\mathcal{H}(\phi)$  is Sperner. Similarly, for a prime monotone DNF formula  $\psi$  we define the hypergraph  $\mathcal{H}(\psi)$  in a similar way as above, with the only difference that hyperedges are associated with terms instead of clauses. Also in this case,  $\mathcal{H}(\psi)$  is a Sperner hypergraph if  $\psi$  is a prime monotone DNF formula.

Let us now focus on the task of computing  $\psi = \text{DNF}(f)$  from  $\varphi$ . Since we have to build a prime DNF formula  $\psi$  equivalent to the monotone CNF  $\varphi$ , all the prime terms of  $\psi$  will be monotone too. Each of the terms of  $\psi$  will contain as (positive) literals a minimal set of variables sufficient to satisfy  $\varphi$  (when the Boolean value **true** is assigned to them). This is tantamount to choose, for every clause of  $\varphi$ , a literal to which we assign **true**. For example, choosing  $x_1$  satisfies clause  $c_1$ , choosing  $x_2$  satisfies  $c_2$ , and  $c_3$  is already satisfied by having chosen  $x_1$ . To satisfy clause  $c_4$  we can choose  $x_4$ , and so we obtain the prime term  $(x_1 \wedge x_2 \wedge x_4)$ , or  $x_3$ , and so we obtain the non-prime term  $(x_1 \wedge x_2 \wedge x_3)$ . By removing  $x_1$  or  $x_2$  from the non-prime term we obtain the two prime terms  $(x_1 \wedge x_3)$  and  $(x_2 \wedge x_3)$ . The reader can easily check that, in fact,  $\psi = \text{DNF}(f) = (x_1 \wedge x_2 \wedge x_4) \vee (x_1 \wedge x_3) \vee (x_2 \wedge x_3)$ . So, to derive the DNF form of  $f$  from its CNF representation, we have “covered” all the clauses of  $\varphi$ , that is, we have essentially evaluated the minimal transversals of  $\mathcal{G} = \mathcal{H}(\varphi)$ . In Figure 4b is reported the hypergraph  $\mathcal{H} = \mathcal{H}(\psi)$ , and the reader can see that  $\mathcal{H}(\psi) = \text{tr}(\mathcal{H}(\varphi))$ .

Computing the prime CNF form of a monotone Boolean function from its prime DNF form can be done in

a similar way. Let us take into consideration again the function  $f$  represented in DNF by

$$\psi = DNF(f) = \underbrace{(x_1 \wedge x_2 \wedge x_4)}_{t_1} \vee \underbrace{(x_1 \wedge x_3)}_{t_2} \vee \underbrace{(x_2 \wedge x_3)}_{t_3}.$$

In order to compute the formula  $\varphi = CNF(f)$  from  $\psi$ , just consider the following. Every clause  $c$  of  $\varphi$  is an implicate of  $f$ , i.e.,  $f \leq c$ , from which it follows that  $\neg c \leq \neg f$ . So, by looking for how to falsify  $f$ , and  $\psi$  in particular, we can compute every single complemented prime clause  $\neg c$  of  $\varphi$ , which, complemented again, allows us to easily obtain the prime clause  $c$ . A way to falsify  $\psi$  is that of choosing, for every term of  $\psi$ , a literal to which we assign **false**. For example, to falsify term  $t_1$  we can choose  $x_1$ , and term  $t_2$  is already falsified by the previous choice of  $x_1$ . To falsify  $t_3$  we can take  $x_2$  or  $x_3$ , hence obtaining  $\neg c_1 = (\neg x_1 \wedge \neg x_2)$  and  $\neg c_3 = (\neg x_1 \wedge \neg x_3)$ , and thus  $c_1 = \neg(\neg c_1) = (x_1 \vee x_2)$ , and  $c_3 = \neg(\neg c_3) = (x_1 \vee x_3)$ . Following this line of reasoning the reader can compute the whole formula  $\varphi = CNF(f) = (x_1 \vee x_2) \wedge (x_2 \vee x_3) \wedge (x_1 \vee x_3) \wedge (x_3 \vee x_4)$ . Also in this case, the way to compute  $\varphi$  from  $\psi$  was essentially that of computing all the minimal transversals of the hypergraph  $\mathcal{H} = \mathcal{H}(\psi)$  (see Figure 4b) to obtain the hypergraph  $\mathcal{G} = \mathcal{H}(\varphi)$  (see Figure 4a). Observe that  $\mathcal{G}$  and  $\mathcal{H}$  are such that  $\mathcal{G} = tr(\mathcal{H})$ .

We have just seen that the transversal hypergraph computation is essentially the same task of computing the DNF prime form of a monotone Boolean function when its CNF prime form is given in input, and vice-versa. To see that this is also related to the computation of the dual function, i.e., the dualization problem properly said, the step is simple. If, from the formula  $\varphi = CNF(f)$ , we derive the formula  $\psi = DNF(f)$  through a procedure that essentially computes  $tr(\mathcal{H}(\varphi))$ , then  $\psi^\uparrow$  is a prime monotone CNF formula such that  $\psi^\uparrow = CNF(f^d)$  (remember that the ‘ $\uparrow$ ’ operator only interchanges  $\wedge$  with  $\vee$ , and 0 with 1, and no negation  $\neg$  operator is introduced in the newly generated formula). Similarly, for DNF formulas, if from  $\psi = DNF(f)$  we obtain  $\varphi = CNF(f)$  (through the computation of  $tr(\mathcal{H}(\psi))$ ), then  $\varphi^\uparrow = DNF(f^d)$ .

Since we will deal with the decision problem version of the transversal hypergraph problem, let us now focus our attention on the duality problem. This problem can be formulated in two equivalent forms: (1) given two prime CNF (or DNF) formulas, decide whether they represent dual Boolean functions; (2) given a prime monotone CNF formula and a prime monotone DNF formula, decide whether these two formulas are equivalent (i.e., they represent the same monotone Boolean function). These problems, for the discussion above, are equivalent to deciding whether two given hypergraphs are each the transversal hypergraph of the other.

To relate our work to the other in the literature, let us see what including or excluding a vertex in our approach means for the other works focused on Boolean formulas. Suppose that  $\mathcal{G}$  and  $\mathcal{H}$  are two hypergraphs, and we want to decide whether  $\mathcal{H} = tr(\mathcal{G})$ . In order to answer “no” to this question, if  $\mathcal{H}$  consists only of transversal of  $\mathcal{G}$ , we need to find a transversal of  $\mathcal{G}$  that is not in  $\mathcal{H}$ . We call such a transversal, a *new transversal of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$* . To do so, inspired also by the algorithm of Gaur [24, 25], it is useful to keep track of two sets. The set of the vertices already included in the ongoing building transversal, and the set of the vertices already excluded from the ongoing building transversal. In this way, it is easy to check what hyperedges of  $\mathcal{G}$  are already covered by and what hyperedges of  $\mathcal{H}$  are already different from the candidate for a new transversal (more details of this will be given in Section 3).

Following the notation of Elbassioni [19] and Boros and Makino [4], given a hypergraph  $\mathcal{G}$  and a set  $S$  of vertices, we define hypergraphs  $\mathcal{G}_S = \langle S, \{G \in \mathcal{G} \mid G \subseteq S\} \rangle$ , and  $\mathcal{G}^S = \langle S, \min(\{G \cap S \mid G \in \mathcal{G}\}) \rangle$ , where  $\min(\mathcal{H})$ , for any hypergraph  $\mathcal{H}$ , denotes the set of inclusion minimal edges of  $\mathcal{H}$ .

Now assume, for example, that we want to include a particular vertex  $v$  in the new transversal of  $\mathcal{G}$ . After the inclusion of  $v$ , all the hyperedges of  $\mathcal{G}$  containing  $v$  are covered and hence they no longer need to be considered in the construction of the new transversal of  $\mathcal{G}$ . So, the new hypergraph “ $\mathcal{G}$ ” (of the pair) to be taken into consideration is  $\mathcal{G}_{V \setminus \{v\}}$ . See, for example, Figure 5a in which is represented the graph  $\mathcal{G}_{V \setminus \{v_1\}}$ , where  $\mathcal{G}$  is the hypergraph of Figure 4a. On the other hand, only the exclusion of a vertex belonging to an edge  $H \in \mathcal{H}$  makes  $H$  certainly different from the ongoing building transversal. In fact, when, on the contrary, vertices are included, the ongoing building transversal could grow to the point to be a superset of  $H$ , and hence the just built transversal would not be new w.r.t.  $\mathcal{H}$ . So, since including a vertex in the new transversal of  $\mathcal{G}$  does not make any of the hyperedges of  $\mathcal{H}$  different from the new candidate one, then all the hyperedges of  $\mathcal{H}$  still need to be considered, apart from removing the included vertex and shrinking the proper hyperedges. That is, the new hypergraph “ $\mathcal{H}$ ” (of the pair) to be considered is  $\mathcal{H}^{V \setminus \{v\}}$ . See Figure 5b in which is depicted the graph  $\mathcal{H}^{V \setminus \{v_1\}}$ , where  $\mathcal{H}$  is the hypergraph of Figure 4b.

We can observe the following connection. Given a pair of hypergraphs  $\langle \mathcal{G}, \mathcal{H} \rangle$ , with the aim of building a new transversal of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$ , in our approach including a vertex  $v_i$  in a candidate new transversal is equivalent to consider the “updated” pair  $\langle \mathcal{G}_{V \setminus \{v_i\}}, \mathcal{H}^{V \setminus \{v_i\}} \rangle$  when we use the notation of Elbassioni [19] and Boros and Makino [4].

Let us link this to the works focused on Boolean formulas, and in particular to the problem of deciding whether a CNF and a DNF formula are equivalent. Consider the CNF formula  $\varphi$  such that  $\mathcal{G} = \mathcal{H}(\varphi)$ , and

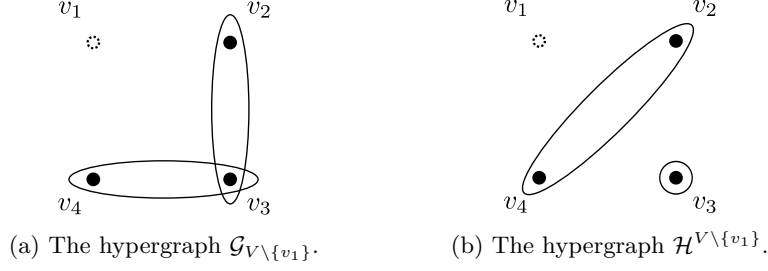


Figure 5: The result of including vertex  $v_1$  in the aim of building a new transversal of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$ . Hypergraphs  $\mathcal{G}$  and  $\mathcal{H}$  are those depicted in Figures 4a and 4b, respectively. The dashed vertices do not belong any longer to the hypergraphs.

the DNF formula  $\psi$  such that  $\mathcal{H} = \mathcal{H}(\psi)$ . Let  $\langle \varphi, \psi \rangle$  be the pair of formulas for which it has to be decided whether  $\varphi$  and  $\psi$  are equivalent. We are going to see that including a vertex  $v_i$  is equivalent to assigning **true** to the variable  $x_i$  related to  $v_i$ , in both  $\varphi$  and  $\psi$ . Indeed, in the formula  $\varphi$ , since it is in CNF, assigning **true** to  $x_i$  means covering all the clauses containing  $x_i$ , and hence they do not need to be considered further (as in  $\mathcal{G}_{V \setminus \{v_i\}}$ ). While assigning **true** to  $x_i$  in  $\psi$ , since it is in DNF, alters its terms containing  $x_i$  by deleting that literal, and does not alter at all the other terms (as in  $\mathcal{H}^{V \setminus \{v_i\}}$ ). Instead, if  $\varphi$  is in DNF, and  $\psi$  in CNF, including  $v_i$  is equivalent to assigning **false** to  $x_i$  in  $\varphi$  and  $\psi$ .

By extension, including a set of vertices  $S$  is equivalent to consider the new pair of hypergraphs  $\langle \mathcal{G}_{V \setminus S}, \mathcal{H}^{V \setminus S} \rangle$ , and to assign **true** (**false**, resp.) to all the variables whose related vertices are in  $S$  if  $\varphi$  is in CNF (DNF, resp.), and  $\psi$  is in DNF (CNF, resp.).

Similarly, excluding a set of vertices  $S$  from the new transversal is equivalent to consider the new pair of hypergraphs  $\langle \mathcal{G}^{V \setminus S}, \mathcal{H}_{V \setminus S} \rangle$ , and to assign **false** (**true**, resp.) to all the variables whose related vertices are in  $S$  if  $\varphi$  is in CNF (DNF, resp.), and  $\psi$  is in DNF (CNF, resp.).

By combining the two observations above, when  $In$  is a set of included vertices, and  $Ex$  is a set of excluded vertices, the updated pair is  $\langle (\mathcal{G}_{V \setminus In})^{V \setminus (In \cup Ex)}, (\mathcal{H}_{V \setminus Ex})^{V \setminus (In \cup Ex)} \rangle$ . This expression will be used in Section 3.

When we consider the problem of deciding whether two formulas are dual, the link is as follows. Consider the two CNF (DNF, resp.) formulas  $\varphi$  and  $\psi$  such that  $\mathcal{G} = \mathcal{H}(\varphi)$  and  $\mathcal{H} = \mathcal{H}(\psi)$ . Let  $\langle \varphi, \psi \rangle$  be the pair of formulas for which it has to be decided whether  $\varphi$  and  $\psi$  are dual. Including a vertex  $v_i$  in a candidate new transversal of  $\mathcal{G}$  is equivalent to assign to  $x_i$  **true** (**false**, resp.) in  $\varphi$  and **false** (**true**, resp.) in  $\psi$ . While excluding  $v_i$  from the candidate new transversal of  $\mathcal{G}$  is equivalent to assign to  $x_i$  **false** (**true**, resp.) in  $\varphi$  and **true** (**false**, resp.) in  $\psi$ . These considerations can be generalized in the obvious way to the inclusion and exclusion of sets of vertices.

## C Proofs of properties stated in Section 2

**Lemma 2.1.** *Let  $\mathcal{H}$  be a hypergraph, and  $T \subseteq V$  be a transversal of  $\mathcal{H}$ . Then,  $T$  is a minimal transversal of  $\mathcal{H}$  if and only if every vertex  $v \in T$  is critical (and hence there exists an edge  $H_v \in \mathcal{H}$  witnessing so).*

*Proof.*

( $\Rightarrow$ ) Let  $T$  be a minimal transversal of  $\mathcal{H}$ , and assume by contradiction that there exists a vertex  $v \in T$  that is not critical. This means that, for every edge  $H \in \mathcal{H}$ , if  $v \in (H \cap T)$ , then  $|H \cap T| \geq 2$ . Now consider the set  $T' = T \setminus \{v\}$ , then  $|H \cap T'| \geq 1$  for every edge  $H$  of  $\mathcal{H}$ . Therefore the set  $T' \subset T$  is a transversal of  $\mathcal{H}$ : a contradiction, because we are assuming that  $T$  is a minimal transversal of  $\mathcal{H}$ .

( $\Leftarrow$ ) Assume that all the vertices of the transversal  $T$  are critical. Consider any set  $T' \subset T$  and let  $v \in T \setminus T'$ . Since  $v$  is critical, there exists an edge  $H_v \in \mathcal{H}$  such that  $T \cap H_v = \{v\}$ . Hence,  $T' \cap H_v = \emptyset$  and  $T'$  is not a transversal of  $\mathcal{H}$ . Therefore  $T$  is a minimal transversal of  $\mathcal{H}$ .  $\square$

**Lemma 2.2.** *Let  $\mathcal{G}$  and  $\mathcal{H}$  be two hypergraphs. A set of vertices  $T \subseteq V$  is a new transversal of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$  if and only if  $\overline{T}$  is a new transversal of  $\mathcal{H}$  w.r.t.  $\mathcal{G}$ .*

*Proof.* If  $T$  is a new transversal of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$ , then  $T$  is an independent set of  $\mathcal{H}$ . This means that for every edge  $H \in \mathcal{H}$  there is a vertex  $v \in H \setminus T$ . Hence  $v \in \overline{T}$ , and so  $\overline{T}$  is a transversal of  $\mathcal{H}$ . Moreover, the set  $\overline{T}$  cannot include any edge of  $\mathcal{G}$ , for otherwise  $T$  would not intersect that edge of  $\mathcal{G}$ , and  $T$  would not be a transversal of  $\mathcal{G}$ . Therefore,  $\overline{T}$  is an independent set of  $\mathcal{G}$ , which proves that  $\overline{T}$  is a new transversal of  $\mathcal{H}$  w.r.t.  $\mathcal{G}$ . For the

other direction, just swap the roles of  $\mathcal{G}$  and  $\mathcal{H}$ . Therefore, if  $\overline{T}$  is a new transversal of  $\mathcal{H}$ , then  $\overline{\overline{T}} = T$  is a new transversal of  $\mathcal{G}$ .  $\square$

**Lemma 2.3.** *Let  $\mathcal{G}$  and  $\mathcal{H}$  be two hypergraphs. Then,  $\mathcal{G}$  and  $\mathcal{H}$  are dual if and only if  $\mathcal{G}$  and  $\mathcal{H}$  are simple, satisfy the intersection property, and there is no new transversal of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$ .*

*Proof.*

( $\Rightarrow$ ) If  $\mathcal{G}$  and  $\mathcal{H}$  are dual, then they must be simple. Indeed, assume by contradiction that  $\mathcal{G}$  is not simple. Then there are two different edges  $G', G'' \in \mathcal{G}$  such that  $G' \subset G''$ . There are two cases: either  $G'$  is a transversal of  $\mathcal{H}$ , or it is not.

If  $G'$  is a transversal of  $\mathcal{H}$ , then  $G''$  is not a minimal transversal of  $\mathcal{H}$ , and hence  $\mathcal{H} \neq tr(\mathcal{G})$ : a contraction, because we are assuming  $\mathcal{G}$  and  $\mathcal{H}$  to be dual. On the other hand, if  $G'$  is not a transversal of  $\mathcal{H}$ , then, again,  $\mathcal{H} \neq tr(\mathcal{G})$ : a contraction, because we are assuming  $\mathcal{G}$  and  $\mathcal{H}$  to be dual. Similarly it can be shown that  $\mathcal{H}$  must be simple.

Assume now by contradiction that  $\mathcal{G}$  and  $\mathcal{H}$  do not satisfy the intersection property. Then there are edges  $G \in \mathcal{G}$  and  $H \in \mathcal{H}$  such that  $G \cap H = \emptyset$ : a contradiction, because we are assuming  $\mathcal{G}$  and  $\mathcal{H}$  to be dual.

Assume now by contradiction that there is a new transversal  $T$  of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$ . Let  $T' \subseteq T$  be any minimal transversal of  $\mathcal{G}$ . Being  $T$  an independent set of  $\mathcal{H}$ , then so is  $T'$ . This implies that, for all the edges  $H \in \mathcal{H}$ , there exists a vertex  $v$  such that  $v \in H \setminus T'$ , and hence  $H \neq T'$ . Thus,  $T'$  is a minimal transversal of  $\mathcal{G}$  missing in  $\mathcal{H}$ , and hence  $\mathcal{H} \neq tr(\mathcal{G})$ : a contradiction, because we are assuming  $\mathcal{G}$  and  $\mathcal{H}$  to be dual.

( $\Leftarrow$ ) Since there is no new transversal of  $\mathcal{G}$ , for every set of vertices  $T \subseteq V$ , the set  $T$  is not a transversal of  $\mathcal{G}$  or it is not an independent set of  $\mathcal{H}$  (or both). Consider the case in which  $T$  is any of the minimal transversals of  $\mathcal{G}$ . In this case  $T$  is not an independent set of  $\mathcal{H}$  (because no new transversal of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$  exists), and hence there exists an edge  $H \in \mathcal{H}$  such that  $H \subseteq T$ . Since all the edges of  $\mathcal{H}$  are transversals of  $\mathcal{G}$  (because  $\mathcal{G}$  and  $\mathcal{H}$  satisfy the intersection property), we claim that  $H = T$ . Indeed, since  $H \subseteq T$  and  $T$  is a minimal transversal of  $\mathcal{G}$ , being  $H$  a transversal of  $\mathcal{G}$  it cannot happen that  $H$  is strictly contained in  $T$ , hence  $H = T$  and thus  $tr(\mathcal{G}) \subseteq \mathcal{H}$ . Moreover, because  $\mathcal{H}$  is a simple hypergraph it must be the case that  $\mathcal{H} = tr(\mathcal{G})$ . Thus, hypergraphs  $\mathcal{G}$  and  $\mathcal{H}$  are dual.  $\square$

## D Proofs of properties stated in Section 3

**Lemma 3.1.** *Let  $\mathcal{G}$  and  $\mathcal{H}$  be two hypergraphs, and let  $\sigma = \langle In, Ex \rangle$  be an assignment.*

- (a) *If  $T$  is a transversal of  $\mathcal{G}$  coherent with  $\sigma$ , then  $Ex$  is not covering;*
- (b) *If  $T$  is an independent set of  $\mathcal{H}$  coherent with  $\sigma$ , then  $In$  is not covering.*

*Hence, if  $T$  is a new transversal of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$  coherent with  $\sigma$ , then  $\sigma$  is not covering.*

*Proof.*

- (a) Assume by contradiction that exists an edge  $G \in \mathcal{G}$  such that  $G \subseteq Ex$ . Since  $T$  is a transversal of  $\mathcal{G}$ ,  $T \cap Ex \neq \emptyset$ : a contradiction, because  $\sigma$  is coherent with  $T$ .
- (b) Since  $\sigma$  is coherent with  $T$ ,  $In \subseteq T$ . Hence, from  $T$  being an independent set of  $\mathcal{H}$  it follows that  $In$  is an independent set of  $\mathcal{H}$  (i.e.,  $In$  is not covering).  $\square$

**Lemma D.1.** *Let  $\mathcal{G}$  and  $\mathcal{H}$  be two hypergraphs, and let  $\sigma = \langle In, Ex \rangle$  be an assignment. Then:*

- (a)  $\mathcal{G}_{V \setminus In} \neq \{\emptyset\}$ ; and,  
 $\mathcal{H}_{V \setminus Ex} \neq \{\emptyset\}$ ;
- (b)  $\mathcal{G}_{V \setminus In} = \emptyset$  if and only if  $In$  is a transversal of  $\mathcal{G}$ ; and,  
 $\mathcal{H}_{V \setminus Ex} = \emptyset$  if and only if  $Ex$  is a transversal of  $\mathcal{H}$ ;
- (c)  $\mathcal{G}(\sigma) = \emptyset$  if and only if  $\mathcal{G}_{V \setminus In} = \emptyset$  (and hence if and only if  $In$  is a transversal of  $\mathcal{G}$ ); and,  
 $\mathcal{H}(\sigma) = \emptyset$  if and only if  $\mathcal{H}_{V \setminus Ex} = \emptyset$  (and hence if and only if  $Ex$  is a transversal of  $\mathcal{H}$ );
- (d)  $\mathcal{G}(\sigma) = \{\emptyset\}$  if and only if  $In$  is not a transversal of  $\mathcal{G}$  and  $Ex$  is covering; and,  
 $\mathcal{H}(\sigma) = \{\emptyset\}$  if and only if  $Ex$  is not a transversal of  $\mathcal{H}$  and  $In$  is covering;

- (e)  $\mathcal{G}(\sigma)$  contains non-empty edges if and only if  $In$  is not a transversal of  $\mathcal{G}$  and  $Ex$  is not covering; and,  $\mathcal{H}(\sigma)$  contains non-empty edges if and only if  $Ex$  is not a transversal of  $\mathcal{H}$  and  $In$  is not covering.

*Proof.*

- (a) By definition, either  $\mathcal{G}_{V \setminus In} = \emptyset$ , or  $\mathcal{G}_{V \setminus In}$  contains non-empty edges (it cannot be the case that  $\mathcal{G}_{V \setminus In} = \{\emptyset\}$ ). Similarly, it can be proven that  $\mathcal{H}_{V \setminus Ex} \neq \{\emptyset\}$ .
- (b) The property is true by definitions of  $\mathcal{G}_{V \setminus In}$  and  $\mathcal{H}_{V \setminus Ex}$ .
- (c) We prove the property for  $\mathcal{G}(\sigma)$ . The proof for  $\mathcal{H}(\sigma)$  is symmetric.
- ( $\Rightarrow$ )  $\mathcal{G}_{V \setminus In} \neq \{\emptyset\}$  by point (a). Assume by contradiction that  $\mathcal{G}_{V \setminus In}$  contains non-empty edges, and let  $G \in \mathcal{G}_{V \setminus In}$ . If  $G \cap (V \setminus (In \cup Ex))$  is empty, then  $\mathcal{G}(\sigma) = \{\emptyset\}$ : a contradiction, because we are assuming  $\mathcal{G}(\sigma) = \emptyset$ . On the other hand, if  $G \cap (V \setminus (In \cup Ex))$  is not empty, then  $\mathcal{G}(\sigma)$  contains non-empty edges: a contradiction, because we are assuming  $\mathcal{G}(\sigma) = \emptyset$ . Therefore, it must be the case that  $\mathcal{G}_{V \setminus In} = \emptyset$ .
- ( $\Leftarrow$ ) If  $\mathcal{G}_{V \setminus In} = \emptyset$ , then, by definition,  $\mathcal{G}(\sigma) = \emptyset$ .
- (d) We prove the property for  $\mathcal{G}(\sigma)$ . The proof for  $\mathcal{H}(\sigma)$  is symmetric.
- ( $\Rightarrow$ ) If  $\mathcal{G}(\sigma) = \{\emptyset\}$ , then, by points (c) and (b),  $In$  is not a transversal of  $\mathcal{G}$ . By points (c) and (a),  $\mathcal{G}_{V \setminus In}$  contains non-empty edges. Assume by contradiction that  $Ex$  is not covering, and hence, since  $\mathcal{G}_{V \setminus In} \subseteq \mathcal{G}$ , each edge  $G \in \mathcal{G}_{V \setminus In}$  is such that  $G \not\subseteq Ex$ . By definition of  $\mathcal{G}_{V \setminus In}$ ,  $G \cap In = \emptyset$ , and, since there is a vertex  $v \in (G \setminus Ex)$ ,  $G \cap (V \setminus (In \cup Ex)) \neq \emptyset$ . Therefore,  $\mathcal{G}(\sigma) \neq \{\emptyset\}$ : a contradiction, because we are assuming  $\mathcal{G}(\sigma) = \{\emptyset\}$ . Thus there is an edge  $G \in \mathcal{G}_{V \setminus In}$  such that  $G \subseteq Ex$ . To conclude, observe that, by  $\mathcal{G}_{V \setminus In} \subseteq \mathcal{G}$ ,  $G \in \mathcal{G}$  too, and hence  $Ex$  is covering.
- ( $\Leftarrow$ ) By points (b) and (a), since  $In$  is not a transversal of  $\mathcal{G}$ ,  $\mathcal{G}_{V \setminus In}$  contains non-empty edges, and in particular  $\mathcal{G}_{V \setminus In}$  contains all and only the edges  $G \in \mathcal{G}$  such that  $G \cap In = \emptyset$ . Since  $Ex$  is covering, there is an edge  $G \in \mathcal{G}$  such that  $G \subseteq Ex$ , and observe that  $G$  must belong to  $\mathcal{G}_{V \setminus In}$  (because  $In$  and  $Ex$  are disjoint). Therefore, from  $G \in \mathcal{G}_{V \setminus In}$  and  $G \cap (V \setminus (In \cup Ex)) = \emptyset$ , it follows that  $\mathcal{G}(\sigma) = \{\emptyset\}$ .
- (e) The validity of this point directly descends from points (c) and (d).  $\square$

**Lemma D.2.** *Let  $\mathcal{G}$  and  $\mathcal{H}$  be two hypergraphs satisfying the intersection property, and let  $\sigma = \langle In, Ex \rangle$  be a covering assignment. Then:*

- (a)  $In$  and  $Ex$  cannot be both covering;
- (b)  $\mathcal{G}(\sigma)$  and  $\mathcal{H}(\sigma)$  are trivially dual. In particular, if  $In$  is covering, then  $\mathcal{G}(\sigma) = \emptyset$  and  $\mathcal{H}(\sigma) = \{\emptyset\}$ , and, symmetrically, if  $Ex$  is covering, then  $\mathcal{G}(\sigma) = \{\emptyset\}$  and  $\mathcal{H}(\sigma) = \emptyset$ .

*Proof.*

- (a) Assume by contradiction that there are two edges  $G \in \mathcal{G}$  and  $H \in \mathcal{H}$  such that  $G \subseteq Ex$  and  $H \subseteq In$ . Because of the intersection property, from  $G \cap H \neq \emptyset$  follows  $In \cap Ex \neq \emptyset$ : a contradiction, because  $\sigma$  is an assignment.
- (b) If  $In$  is covering, then  $In$  is a transversal of  $\mathcal{G}$  because  $\mathcal{G}$  and  $\mathcal{H}$  satisfy the intersection property. Thus,  $\mathcal{G}(\sigma) = \emptyset$  (see Lemma D.1). Since  $In$  is covering, from  $In \cap Ex = \emptyset$  it follows that  $Ex$  is not a transversal of  $\mathcal{H}$ . Hence, by Lemma D.1,  $\mathcal{H}(\sigma) = \{\emptyset\}$ . Therefore  $\mathcal{G}(\sigma)$  and  $\mathcal{H}(\sigma)$  are trivially dual. Symmetrically, if  $Ex$  is covering, then  $\mathcal{H}(\sigma) = \emptyset$  and  $\mathcal{G}(\sigma) = \{\emptyset\}$ . So, again,  $\mathcal{G}(\sigma)$  and  $\mathcal{H}(\sigma)$  are trivially dual.  $\square$

**Lemma D.3.** *Let  $\mathcal{G}$  and  $\mathcal{H}$  be two hypergraphs satisfying the intersection property, and let  $\sigma$  be an assignment. Then,  $\mathcal{G}(\sigma)$  and  $\mathcal{H}(\sigma)$  satisfy the intersection property.*

*Proof.* Let  $\sigma = \langle In, Ex \rangle$ . If  $\sigma$  is covering, then, by Lemma D.2,  $\mathcal{G}(\sigma)$  and  $\mathcal{H}(\sigma)$  are dual, and hence they satisfy also the intersection property.

In case  $\sigma$  is non-covering, if  $In$  (resp.  $Ex$ ) is a transversal of  $\mathcal{G}$  (resp.  $\mathcal{H}$ ), then  $\mathcal{G}(\sigma) = \emptyset$  (resp.  $\mathcal{H}(\sigma) = \emptyset$ ) (see Lemma D.1). In these cases, since at least one of the two hypergraphs  $\mathcal{G}(\sigma)$  and  $\mathcal{H}(\sigma)$  is empty, they satisfy the intersection property.

Let us consider now the case in which both  $In$  and  $Ex$  are not transversals of  $\mathcal{G}(\sigma)$  and  $\mathcal{H}(\sigma)$ , respectively. By Lemma D.1, both  $\mathcal{G}(\sigma)$  and  $\mathcal{H}(\sigma)$  contain non-empty edges. Assume by contradiction that there exist two edges  $G' \in \mathcal{G}(\sigma)$  and  $H' \in \mathcal{H}(\sigma)$  such that  $G' \cap H' = \emptyset$ . Since  $G' \in \mathcal{G}(\sigma)$  there exists an edge  $G \in \mathcal{G}$  such that  $G = G' \cup A$ , with  $\emptyset \subseteq A \subseteq Ex$ , and with  $G \cap In = \emptyset$  (for otherwise  $G'$  would not be in  $\mathcal{G}(\sigma)$ ). Moreover, since  $H' \in \mathcal{H}(\sigma)$  there exists an edge  $H \in \mathcal{H}$  such that  $H = H' \cup B$ , with  $\emptyset \subseteq B \subseteq In$ , and with  $H \cap Ex = \emptyset$  (for otherwise  $H'$  would not be in  $\mathcal{H}(\sigma)$ ). From this follows  $G \cap H = \emptyset$ : a contradiction, because  $\mathcal{G}$  and  $\mathcal{H}$  satisfy the intersection property. Therefore,  $\mathcal{G}(\sigma)$  and  $\mathcal{H}(\sigma)$  satisfy the intersection property.  $\square$



**Lemma D.4.** *Let  $\mathcal{G}$  and  $\mathcal{H}$  be two hypergraphs, and let  $\sigma = \langle In, Ex \rangle$  be an assignment.*

- (a) *If  $T'$  is a transversal of  $\mathcal{G}(\sigma)$ , then  $T = T' \cup In$  is a transversal of  $\mathcal{G}$ ;*
- (b) *If  $T'$  is an independent set of  $\mathcal{H}(\sigma)$ , then  $T = T' \cup In$  is an independent set of  $\mathcal{H}$ .*

*Hence, if  $T'$  is a new transversal of  $\mathcal{G}(\sigma)$  w.r.t.  $\mathcal{H}(\sigma)$ , then  $T = T' \cup In$  is a new transversal of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$ .*

*Proof.*

- (a) If  $\mathcal{G}(\sigma) = \{\emptyset\}$ , then there are no transversals of  $\mathcal{G}(\sigma)$  at all, so we can focus on  $\mathcal{G}(\sigma) \neq \{\emptyset\}$ . Observe that, for each edge  $G \in \mathcal{G}$ , there are two cases: either  $G \cap In \neq \emptyset$  or  $G \cap In = \emptyset$ . If  $G \cap In \neq \emptyset$ , then  $T \cap G \neq \emptyset$  because  $In \subseteq T$ . If  $G \cap In = \emptyset$ , then there is an edge  $\emptyset \neq G' \in \mathcal{G}(\sigma)$  such that  $G' \subseteq G$ . Hence,  $T \cap G \neq \emptyset$  because  $T' \cap G' \neq \emptyset$ ,  $G' \subseteq G$ , and  $T' \subseteq T$ .
- (b) If  $\mathcal{H}(\sigma) = \{\emptyset\}$ , then there are no independent sets of  $\mathcal{H}(\sigma)$  at all, so we can focus on  $\mathcal{H}(\sigma) \neq \{\emptyset\}$ . Since  $T'$  is an independent set of  $\mathcal{H}(\sigma)$ ,  $T' \subseteq V \setminus (In \cup Ex)$ . Observe that, for each edge  $H \in \mathcal{H}$ , there are two cases: either  $H \cap Ex \neq \emptyset$  or  $H \cap Ex = \emptyset$ . From  $T' \subseteq V \setminus (In \cup Ex)$  and  $In \cap Ex = \emptyset$ , it follows that  $T \cap Ex = \emptyset$ . So, if  $H \cap Ex \neq \emptyset$ , then  $H \not\subseteq T$ . On the other hand, if  $H \cap Ex = \emptyset$ , then there is an edge  $\emptyset \neq H' \in \mathcal{H}(\sigma)$  such that  $H' \subseteq H$ . Since  $T'$  is an independent set of  $\mathcal{H}(\sigma)$ , there exists a vertex  $v \in (H' \setminus T')$ . From  $H' \subseteq (V \setminus (In \cup Ex))$  and  $v \in H'$ , it follows that  $v \notin In$ . Therefore, since  $T = T' \cup In$  and  $H' \subseteq H$ ,  $H \not\subseteq T$ .  $\square$

Observe that, by the symmetrical nature of the DUAL problem, the roles of  $\mathcal{G}$  and  $\mathcal{H}$  can be swapped in an instance of DUAL. By this reason, Lemma D.4 can be easily adapted to state that transversals of  $\mathcal{H}(\sigma)$  and independent sets of  $\mathcal{G}(\sigma)$  can be extended, in this case by adding  $Ex$ , to be transversals of  $\mathcal{H}$  and independent sets of  $\mathcal{G}$ , respectively.

**Lemma D.5.** *Let  $\mathcal{G}$  and  $\mathcal{H}$  be two dual hypergraphs. Then, for all assignments  $\sigma$ ,  $\mathcal{G}(\sigma)$  and  $\mathcal{H}(\sigma)$  are dual.*

*Proof.* Since  $\mathcal{G}$  and  $\mathcal{H}$  are dual, they are also simple and satisfy the intersection property (see Lemma 2.3). Let  $\sigma = \langle In, Ex \rangle$ . If  $\sigma$  is covering, then, by Lemma D.2,  $\mathcal{G}(\sigma)$  and  $\mathcal{H}(\sigma)$  are dual.

Consider now the case in which  $\sigma$  is non-covering. If  $In$  were a transversal of  $\mathcal{G}$ , then  $In$  would be a superset of an edge of  $\mathcal{H}$  (i.e.,  $In$  would not be an independent set of  $\mathcal{H}$ ), because  $\mathcal{G}$  and  $\mathcal{H}$  are dual, and hence  $\sigma$  would be covering: a contradiction. Similarly, if  $Ex$  were a transversal of  $\mathcal{H}$ , then  $Ex$  would not be an independent set of  $\mathcal{G}$ , and hence  $\sigma$  would be covering: a contradiction. So we can consider the case in which  $In$  and  $Ex$  are *not* a transversal of  $\mathcal{G}$  and  $\mathcal{H}$ , respectively.

In this case, by Lemma D.1, both  $\mathcal{G}(\sigma)$  and  $\mathcal{H}(\sigma)$  contain non-empty edges, and since  $\mathcal{G}$  and  $\mathcal{H}$  satisfy the intersection property because they are dual, by Lemma D.3,  $\mathcal{G}(\sigma)$  and  $\mathcal{H}(\sigma)$  satisfy the intersection property. Remember that, by definition,  $\mathcal{G}(\sigma)$  and  $\mathcal{H}(\sigma)$  are also simple, hence, by Lemma 2.3,  $\mathcal{G}(\sigma)$  and  $\mathcal{H}(\sigma)$  are dual if and only if there is no new transversal of  $\mathcal{G}(\sigma)$  w.r.t.  $\mathcal{H}(\sigma)$ .

But, if there were a new transversal of  $\mathcal{G}(\sigma)$  w.r.t.  $\mathcal{H}(\sigma)$ , by Lemma D.4, there would be a new transversal of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$ , contradicting their duality. Therefore there is no new transversal of  $\mathcal{G}(\sigma)$  w.r.t.  $\mathcal{H}(\sigma)$ , and hence  $\mathcal{G}(\sigma)$  and  $\mathcal{H}(\sigma)$  are actually dual.  $\square$

**Lemma D.6.** *Let  $\mathcal{G}$  and  $\mathcal{H}$  be two hypergraphs, and let  $\sigma = \langle In, Ex \rangle$  be an assignment.*

- (a) *If  $T$  is a transversal of  $\mathcal{G}$  coherent with  $\sigma$ , then  $T' = T \setminus In$  is a transversal of  $\mathcal{G}(\sigma)$ ;*
- (b) *If  $T$  is an independent set of  $\mathcal{H}$  coherent with  $\sigma$ , then  $T' = T \setminus In$  is an independent set of  $\mathcal{H}(\sigma)$ .*

*Hence, if  $T$  is a new transversal of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$  coherent with  $\sigma$ , then  $T' = T \setminus In$  is a new transversal of  $\mathcal{G}(\sigma)$  w.r.t.  $\mathcal{H}(\sigma)$ .*

*Proof.*

- (a) There are two cases: either  $In$  is a transversal of  $\mathcal{G}$ , or it is not. If  $In$  is a transversal of  $\mathcal{G}$ , then  $\mathcal{G}(\sigma) = \emptyset$  (see Lemma D.1). Hence trivially any set of vertices is a transversal of  $\mathcal{G}(\sigma)$ , and so is  $T'$ . Consider now the case in which  $In$  is not a transversal of  $\mathcal{G}$ . Since  $Ex$  is not covering (see Lemma 3.1), by Lemma D.1,  $\mathcal{G}(\sigma)$  contains non-empty edges. By definition of  $\mathcal{G}(\sigma)$ , for any edge  $G' \in \mathcal{G}(\sigma)$  there is an edge  $G \in \mathcal{G}$  such that  $G \cap In = \emptyset$  (for otherwise  $G'$  would not be in  $\mathcal{G}(\sigma)$ ) and  $G' = G \cap (V \setminus (In \cup Ex))$ . Since  $T$  is a transversal of  $\mathcal{G}$ ,  $T \cap (G \setminus In) \neq \emptyset$ , because  $G \cap In = \emptyset$ , and  $T \cap (G \setminus Ex) \neq \emptyset$ , because  $\sigma \sqsubseteq T$  and so  $T \cap Ex = \emptyset$ . Therefore,  $T \cap (G \setminus (In \cup Ex)) \neq \emptyset$ . Since  $G' = G \cap (V \setminus (In \cup Ex))$ ,  $T' = T \setminus In$  has a non-empty intersection with  $G'$ , and hence  $T'$  is a transversal of  $\mathcal{G}(\sigma)$ .

- (b) There are two cases: either  $Ex$  is a transversal of  $\mathcal{H}$ , or it is not. If  $Ex$  is a transversal of  $\mathcal{H}$ , then  $\mathcal{H}(\sigma) = \emptyset$  (see Lemma D.1). Hence trivially any set of vertices is an independent set of  $\mathcal{H}(\sigma)$ , and so is  $T'$ . Consider now the case in which  $Ex$  is not a transversal of  $\mathcal{H}$ . Since  $In$  is not covering (see Lemma 3.1), by Lemma D.1,  $\mathcal{H}(\sigma)$  contains non-empty edges. By definition of  $\mathcal{H}(\sigma)$ , for any edge  $H' \in \mathcal{H}(\sigma)$  there is an edge  $H \in \mathcal{H}$  such that  $H \cap Ex = \emptyset$  (for otherwise  $H'$  would not be in  $\mathcal{H}(\sigma)$ ) and  $H' = H \cap (V \setminus (In \cup Ex))$ . Since  $T$  is an independent set of  $\mathcal{H}$ , there is a vertex  $v \in (H \setminus T)$  such that  $v \notin Ex$ , because  $H \cap Ex = \emptyset$ , and  $v \notin In$ , because  $\sigma \sqsubseteq T$  and so  $In \subseteq T$ . Therefore, from  $v \notin Ex$  and  $v \notin In$ , it follows that  $v \in H'$  because  $H' = H \cap (V \setminus (In \cup Ex))$ , and from  $v \notin T$  (because  $v \in (H \setminus T)$ ), it follows that  $v \notin T'$ . Hence  $H' \not\subseteq T'$ , and thus  $T'$  is an independent set of  $\mathcal{H}(\sigma)$ .  $\square$

Observe again that, by the symmetrical nature of the DUAL problem, by swapping the roles of  $\mathcal{G}$  and  $\mathcal{H}$ , and by considering the reversed assignment  $\bar{\sigma} = \langle Ex, In \rangle$ , Lemma D.6 can be easily adapted to state that transversals of  $\mathcal{H}$  and independent sets of  $\mathcal{G}$  coherent with  $\bar{\sigma}$  can be shrunk, in this case by removing  $Ex$ , to be transversals of  $\mathcal{H}(\sigma)$  and independent sets of  $\mathcal{G}(\sigma)$ , respectively.

**Lemma 3.2.** *Two hypergraphs  $\mathcal{G}$  and  $\mathcal{H}$  are dual if and only if  $\mathcal{G}$  and  $\mathcal{H}$  are simple, satisfy the intersection property, and, for all assignments  $\sigma$ ,  $\mathcal{G}(\sigma)$  and  $\mathcal{H}(\sigma)$  are dual (or, equivalently, there is no new transversal of  $\mathcal{G}(\sigma)$  w.r.t.  $\mathcal{H}(\sigma)$ ).*

*Proof.*

- ( $\Rightarrow$ ) If  $\mathcal{G}$  and  $\mathcal{H}$  are dual, then, by Lemma 2.3, they are simple and satisfy the intersection property. Moreover, by Lemma D.5, for any assignment  $\sigma$ ,  $\mathcal{G}(\sigma)$  and  $\mathcal{H}(\sigma)$  are dual because  $\mathcal{G}$  and  $\mathcal{H}$  are dual. By definition,  $\mathcal{G}(\sigma)$  and  $\mathcal{H}(\sigma)$  are simple, and by Lemma D.3 they satisfy the intersection property. Therefore, since  $\mathcal{G}(\sigma)$  and  $\mathcal{H}(\sigma)$  are dual, by Lemma 2.3, there is no new transversal of  $\mathcal{G}(\sigma)$  w.r.t.  $\mathcal{H}(\sigma)$ .
- ( $\Leftarrow$ ) If  $\mathcal{G}$  and  $\mathcal{H}$  are not dual, then, by Lemma 2.3, they are not simple, or do not satisfy the intersection property, or there is a new transversal of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$ . If  $\mathcal{G}$  or  $\mathcal{H}$  is not simple, or they do not satisfy the intersection property, then this direction of the lemma trivially follows. Let us assume that  $\mathcal{G}$  and  $\mathcal{H}$  are simple, satisfy the intersection property, and let  $T$  be a new transversal of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$ . Consider any assignment  $\sigma$  coherent with  $T$ . Observe that  $\mathcal{G}(\sigma)$  and  $\mathcal{H}(\sigma)$  are simple by definition, and that they satisfy the intersection property by Lemma D.3. By Lemma D.6, there exists a new transversal of  $\mathcal{G}(\sigma)$  w.r.t.  $\mathcal{H}(\sigma)$ , and hence, by Lemma 2.3  $\mathcal{G}(\sigma)$  and  $\mathcal{H}(\sigma)$  are not dual.  $\square$

## E A deterministic algorithm for DUAL

In this section we propose a deterministic duality algorithm DET-DUAL, which is an extension of that proposed by Gaur [24] (see also Gaur and Krishnamurti [25]). The algorithm DET-DUAL here presented is in somewhat different from Gaur's because the latter checks *self*-duality of a single DNF Boolean formula, while ours verifies duality between two hypergraphs.

Given two hypergraphs  $\mathcal{G}$  and  $\mathcal{H}$ , our algorithm DET-DUAL, like many others, aims at finding a new transversal of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$ . To do so, the algorithm builds up, step after step, by including and excluding vertices, a set of vertices intersecting all the edges of  $\mathcal{G}$  that is different from all the edges of  $\mathcal{H}$  (i.e., a set of vertices that is a transversal of  $\mathcal{G}$  and an independent set of  $\mathcal{H}$ ).

As already discussed, choosing vertices to exclude allows us to decrease the number of the edges of  $\mathcal{H}$  that are not different (yet) from the candidate for a new transversal. In particular, when DET-DUAL excludes specific vertices, this algorithm halves the number of the edges of  $\mathcal{H}$  still needed to be considered.

Let us see the details of the algorithm DET-DUAL. After exhibiting the algorithm, we will formally prove some of its properties, and, amongst them, its correctness. The algorithm DET-DUAL, and more specifically the procedure NEW-TRANSVERSAL which checks the existence of new transversals, uses three sets to keep track of the included, excluded, and free vertices of the currently considered assignment, which are denoted by *Included*, *Excluded*, and *Free*, respectively. From Lemma 4.1, it is evident that, in order to check its Condition (3), we need to know what edges of  $\mathcal{G}$  are separated from and what edges of  $\mathcal{H}$  are still compatible with the assignment (and transversal) under construction. To this purpose, in the algorithm we use the sets denoted by *Sep<sub>G</sub>*, and *Com<sub>H</sub>*, respectively.<sup>11</sup>

The algorithm DET-DUAL is listed as Algorithm 3. The aim of the procedure CHECK-SIMPLE-AND-INTERSECTION is checking that hypergraphs  $\mathcal{G}$  and  $\mathcal{H}$  are simple and satisfy the intersection property.

<sup>11</sup>We could dynamically evaluate these sets during the execution of the algorithm, but for simplicity and readability of the algorithm we prefer to have them explicitly represented.

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**Algorithm 3** A deterministic duality algorithm based on Gaur's.

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1: procedure DET-DUAL( $\mathcal{G}, \mathcal{H}$ )
2:   if  $\neg$ CHECK-SIMPLE-AND-INTERSECTION( $\mathcal{G}, \mathcal{H}$ ) then return false;
3:   return  $\neg$ NEW-TRANSVERSAL( $\mathcal{G}, \mathcal{H}, \emptyset, \emptyset, V$ );

4: procedure NEW-TRANSVERSAL( $\mathcal{G}, \mathcal{H}, Included, Excluded, Free$ )
5:    $Sep_{\mathcal{G}} \leftarrow \{G \in \mathcal{G} \mid G \cap Included = \emptyset\}$ ;
6:    $Com_{\mathcal{H}} \leftarrow \{H \in \mathcal{H} \mid H \cap Excluded = \emptyset\}$ ;
7:   if  $(\exists G)(G \in \mathcal{G} \wedge G \subseteq Excluded) \vee (\exists H)(H \in \mathcal{H} \wedge H \subseteq Included)$  then return false;
8:   if  $Sep_{\mathcal{G}} = \emptyset \vee Com_{\mathcal{H}} = \emptyset$  then return true;
9:    $U \leftarrow \{v \in Free \mid v \text{ belongs to at least half of the edges of } Com_{\mathcal{H}}\}$ ;
10:  for each  $v : v \in U$  do
11:    if NEW-TRANSVERSAL( $\mathcal{G}, \mathcal{H}, Included, Excluded \cup \{v\}, Free \setminus \{v\}$ ) then return true;
12:   $Included \leftarrow Included \cup U$ ;
13:   $Free \leftarrow Free \setminus U$ ;
14:   $Sep_{\mathcal{G}} \leftarrow \{G \in \mathcal{G} \mid G \cap Included = \emptyset\}$ ;
15:  if  $(\exists H)(H \in \mathcal{H} \wedge H \subseteq Included)$  then return false;
16:  if  $Sep_{\mathcal{G}} = \emptyset$  then return true;
17:  for each  $G : G \in Sep_{\mathcal{G}}$  do
18:    for each  $v : v \in G \wedge v \in Free$  do
19:      if NEW-TRANSVERSAL( $\mathcal{G}, \mathcal{H}, Included \cup \{v\}, Excluded \cup (G \setminus \{v\}), Free \setminus G$ ) then return true;
20:  return false;

```

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Note that, for simplicity, the procedure NEW-TRANSVERSAL is meant to be called by value. This means that the parameters passed to the procedure are local copies for each specific recursive call. Therefore any modification to those sets affects only the sets of the call currently executed.

We recall here that two hypergraphs  $\mathcal{G}$  and  $\mathcal{H}$  are dual if and only if they are simple, satisfy the intersection property, and there is no new transversal of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$  (see Lemma 2.3). So, after checking that  $\mathcal{G}$  and  $\mathcal{H}$  are simple and satisfy the intersection property (line 2), it is checked that there is no new transversal of  $\mathcal{G}$ . This is achieved by calling NEW-TRANSVERSAL( $\mathcal{G}, \mathcal{H}, \emptyset, \emptyset, V$ ) at line 3, where the sets *Included* and *Excluded* are set to  $\emptyset$ , and *Free* =  $V$ . This procedure is devised to answer **true** if and only if it finds a new transversal of  $\mathcal{G}$  coherent with the assignment on which it is executed. Given a pair of hypergraphs  $\langle \mathcal{G}, \mathcal{H} \rangle$ , and an assignment  $\pi = \langle In, Ex \rangle$ , we say that the procedure NEW-TRANSVERSAL is executed on  $\pi$  whenever it is called with the following parameters: NEW-TRANSVERSAL( $\mathcal{G}, \mathcal{H}, In, Ex, V \setminus (In \cup Ex)$ ). For notational convenience we denote it as NEW-TRANSVERSAL( $\mathcal{G}, \mathcal{H}, \pi$ ), or even more simply NEW-TRANSVERSAL( $\pi$ ) when it is clear from the context what the two hypergraphs  $\mathcal{G}$  and  $\mathcal{H}$  are.

Let us now analyze intuitively the execution of the procedure NEW-TRANSVERSAL. Consider a generic call of the procedure executed on the pair of hypergraphs  $\langle \mathcal{G}, \mathcal{H} \rangle$  and on the assignment  $\pi$ . At line 7 we check whether  $\pi$  is a covering assignment, i.e.,  $Mis(\pi) \neq \emptyset$  or  $Cov(\pi) \neq \emptyset$ , in which case obviously there is no new transversal of  $\mathcal{G}$  coherent with  $\pi$  (see Lemma 4.1). If this is not the case, then the procedure checks (at line 8) whether  $\pi$  is already a witness. Then the procedure computes a set  $U$  (line 9). This set is locally computed, and hence, for the following discussion, let us call it  $U_{\pi}$ . We use the subscript  $\pi$  because the set  $U$  depends on the history of the recursive calls having led to the one currently being executed, and hence depends on the currently considered assignment  $\pi$  having been built so far (and encoded in the sets *Included* and *Excluded*).

The set  $U_{\pi}$  is the set of the free frequent vertices of  $\pi$ . At first, the procedure tries to exclude individually each of the vertices of  $U_{\pi}$  (lines 10–11). If none of these attempts results in the construction of a witness (all the tests performed at line 11 return **false**), all the vertices of  $U_{\pi}$  are included (lines 12–13). Let us call  $\sigma$  the assignment resulting after the inclusion of the vertices of  $U_{\pi}$ . Then, the procedure checks again whether  $\sigma$  is a covering assignment (line 15). Otherwise, the procedure tests whether  $\sigma$  is a witness (line 16). If this is not the case, then the procedure tries to include each of the free vertices of  $\sigma$  as a critical vertex with an edge of  $Sep(\sigma)$  witnessing its criticality (lines 17–19). If for none of these attempts it is possible to find a new transversal of  $\mathcal{G}$  (all the tests performed at line 19 return **false**), then the procedure answers **false** at line 20, meaning that there is no new transversal of  $\mathcal{G}$  coherent with  $\pi$ .

Let us now make some observations on the procedure NEW-TRANSVERSAL. We are going to see that, throughout the whole execution of the procedure and its recursive calls, the sets *Included*, *Excluded*, and *Free*, are always a partition of the set of vertices  $V$ , and hence *Included* and *Excluded* constitute a consistent

assignment (i.e., *Included* and *Excluded* do not overlap). Indeed, a vertex  $v$  excluded at line 11 is taken from the set  $U_\pi$  that is a subset of *Free*. The fact that vertices to be included are taken from  $U_\pi$  is also at the base of the consistency of the operations performed at lines 12–13. For the assignment passed to the recursive calls at line 19 we need a slightly more involved discussion. The edges  $G$  taken into consideration at lines 17–19 belong to  $Sep_{\mathcal{G}}$ , this means that, when the procedure reaches the execution of line 19, none of the vertices of  $G$  is considered included in the currently examined assignment. The edge  $G$  could have a non-empty intersection with *Excluded*, but we know that  $G$  is not totally contained within the set *Excluded* because the test performed at line 7 failed, and hence there is no edge of  $\mathcal{G}$  totally contained in *Excluded*. This implies that any edge  $G$ , taken into consideration at that stage of the execution (line 19), contains always at least a free vertex. So, the considered assignments  $\pi' = \langle In', Ex' \rangle = \sigma + \langle \{v\}, G \setminus \{v\} \rangle$ , for a free vertex  $v$ , passed to the recursive call, is consistent (i.e., the sets *In'* and *Ex'* are not overlapping).

Besides this, every recursive call performed by NEW-TRANSVERSAL( $\pi$ ) is executed on an assignment  $\pi'$  such that  $\pi \sqsubset \pi'$ . This implies that the set of free vertices becomes smaller and smaller from one recursive call to the next. As a result, since the procedure picks from the set of free vertices only those that it tries to include or exclude, every recursion path traversed by NEW-TRANSVERSAL( $\pi$ ) is finite. This is because at some recursion level the set of free vertices is empty.

Regarding the tests performed at lines 5–8, and at lines 14–16, they essentially check Condition (3) of Lemma 4.1. In particular, is tested the condition  $Mis(\sigma) = \emptyset \wedge Cov(\sigma) = \emptyset \wedge (Sep(\sigma) = \emptyset \vee Com(\sigma) = \emptyset)$  which is equivalent. Note that at lines 14–16 the just mentioned condition is not explicitly tested, but are tested those sub-conditions involving only the sets *Included* and  $Sep_{\mathcal{G}}$ . At that point of the execution is reasonable to do so because those sets are the only ones having changed after the previous check of the mentioned condition performed at lines 5–8.

We can now formally prove the correctness of the algorithm DET-DUAL.

**Theorem E.1.** *Let  $\mathcal{G}$  and  $\mathcal{H}$  be two hypergraphs. The call DET-DUAL( $\mathcal{G}, \mathcal{H}$ ) outputs **true** if and only if  $\mathcal{G}$  and  $\mathcal{H}$  are dual.*

In order to prove Theorem E.1 we need some intermediate lemmas. The following property is at the base of the decomposition used in the Algorithm “A” of Fredman and Khachiyan [22]. We state it in a form appropriate for our own subsequent discussion.

**Lemma E.2.** *Let  $\mathcal{G}$  and  $\mathcal{H}$  be two hypergraphs, let  $\sigma$  be an assignment, and let  $v$  be a free vertex in  $\sigma$ . Then, there exists a new transversal of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$  coherent with  $\sigma$  if and only if  $\sigma + \langle \{v\}, \emptyset \rangle$  or  $\sigma + \langle \emptyset, \{v\} \rangle$  is coherent with a new transversal of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$ .<sup>12</sup>*

*Proof.* Let  $T$  be a new transversal of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$  coherent with  $\sigma$ . If  $v \in T$ , then  $T$  is coherent with  $\sigma + \langle \{v\}, \emptyset \rangle$ , symmetrically if  $v \notin T$ , then  $\sigma + \langle \emptyset, \{v\} \rangle$  is coherent with  $T$ . The other direction of the proof is obvious.  $\square$

On the other hand, the following property was used by Gaur in his algorithm [24, 25]. Again, we state it in a form appropriate for our own discussion. (Note the difference with Lemma 3.5. In this case the assignment  $\sigma$  mentioned in the statement of the following lemma is a precursor of a new transversal that is *not* required to be minimal.)

**Lemma E.3.** *Let  $\mathcal{G}$  and  $\mathcal{H}$  be two hypergraphs, and let  $\sigma$  be a precursor of a new (not necessarily minimal) transversal  $T$  of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$ . Then, there exist an edge  $G \in Sep(\sigma)$  and a free vertex  $v \in G$ , such that  $\sigma + \langle \{v\}, G \setminus \{v\} \rangle$  is coherent with a new transversal  $T'$  of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$ , and  $T' \subseteq T$ .*

*Proof.* Let  $T$  be a new transversal of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$ , and  $\sigma = \langle In, Ex \rangle$  be a precursor of  $T$ . Consider an edge  $\hat{G} \in \arg \min_{G \in Sep(\sigma)} \{|G \cap T|\}$ , and let  $v$  be any of the free vertices belonging to  $\hat{G} \cap T$ .

We are going to show that such an edge  $\hat{G}$  and such a vertex  $v$  are well defined. Indeed, since  $\sigma$  is a precursor of  $T$ ,  $Sep(\sigma) \neq \emptyset$  (for otherwise  $\sigma$  would be a witness, and hence not a precursor), and  $Mis(\sigma) = \emptyset$  because  $\sigma$  is coherent with  $T$  (see Lemma 3.1). This implies that there exist edges  $G \in Sep(\sigma)$ , each of them containing at least a vertex  $v$  such that  $v \notin In$  and  $v \notin Ex$ .

Moreover, by the fact that  $\hat{G} \in Sep(\sigma)$ , and hence that  $\hat{G} \cap In = \emptyset$ , it follows that  $\sigma + \langle \{v\}, \hat{G} \setminus \{v\} \rangle$  is actually a consistent assignment (i.e., the sets of included and excluded vertices do not overlap).

Let  $T' = T \setminus (\hat{G} \setminus \{v\})$  (see Figure 6). By definition,  $T'$  is coherent with  $\sigma + \langle \{v\}, \hat{G} \setminus \{v\} \rangle$ . We now claim that  $T'$  is a transversal of  $\mathcal{G}$ . Assume by contradiction that it is not. Then there exists an edge  $\tilde{G} \in \mathcal{G}$  such that  $T' \cap \tilde{G} = \emptyset$  (see Figure 6). From  $In \subset T$  (because  $\sigma$  is a precursor of  $T$ ) and  $\hat{G} \cap In = \emptyset$  follows  $In \subset T'$ . This, with  $T' \cap \tilde{G} = \emptyset$ , implies that  $\tilde{G} \in Sep(\sigma)$ . Since  $T \cap \tilde{G} \neq \emptyset$  (because  $T$  is a transversal of  $\mathcal{G}$ ) and  $T' \cap \tilde{G} = \emptyset$ , by

<sup>12</sup>This Lemma essentially states that if there are new transversals of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$  coherent with  $\sigma$ , then they include or exclude the free vertex  $v$ . (If there is no new transversal either including or excluding  $v$ , then there is no new transversal coherent with  $\sigma$  at all.)

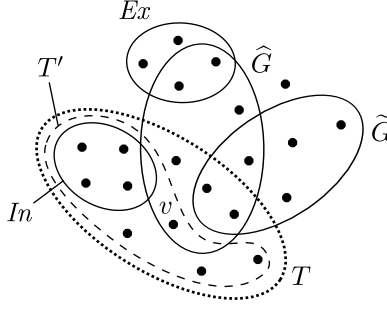


Figure 6: Illustration for Lemma E.3.

the definition of  $T'$  it follows that  $(\tilde{G} \cap T) \subseteq ((\hat{G} \setminus \{v\}) \cap T)$ . For this reason,  $|\tilde{G} \cap T| \leq |(\hat{G} \setminus \{v\}) \cap T| < |\hat{G} \cap T|$  (because  $v \in \hat{G} \cap T$ ): a contradiction, because  $\hat{G}$  was chosen as one of the edges in  $\text{Sep}(\sigma)$  minimizing the size of its intersection with  $T$ . To conclude, note that  $T' \subseteq T$  by definition, and hence  $T'$  is a new transversal of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$  because  $T$ , being a new transversal of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$ , is an independent set of  $\mathcal{H}$ .  $\square$

We now focus on the properties of the procedure **NEW-TRANSVERSAL**.

**Lemma E.4.** *Let  $\mathcal{G}$  and  $\mathcal{H}$  be two hypergraphs, and let  $\pi$  be an assignment coherent with a new transversal of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$ . Then, either **NEW-TRANSVERSAL**( $\pi$ ) answers **true** at line 8 or at line 16, or among its recursive calls there is one executed on an assignment  $\pi'$ , with  $\pi \sqsubset \pi'$ , coherent with a new transversal of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$ .*

*Proof.* Let  $T$  be a new transversal of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$  coherent with  $\pi$ . Since  $\pi \sqsubseteq T$ , and  $T$  is a new transversal of  $\mathcal{G}$ , we know that  $\pi$  cannot be a covering assignment (see Lemma 3.1), and hence **NEW-TRANSVERSAL**( $\pi$ ) cannot return **false** at line 7. If  $\pi$  is already a witness, then **NEW-TRANSVERSAL**( $\pi$ ) answers **true** at line 8 (and the statement of the lemma would be proven).

If this is not the case, then  $\pi$  is only a precursor of  $T$ . Let  $U_\pi$  be the set  $U$  computed by **NEW-TRANSVERSAL**( $\pi$ ) at line 9. At first let us assume that  $U_\pi \neq \emptyset$ . At lines 10–11 recursive calls are performed on the various assignments  $\pi' = \pi + \langle \emptyset, \{v\} \rangle$ , for each vertex  $v \in U_\pi$  (note that  $\pi \sqsubset \pi'$ ). If one of them is coherent with a new transversal of  $\mathcal{G}$ , then the statement of the lemma is proven. If this is not the case, then, by Lemma E.2, the assignment  $\sigma = \pi + \langle U_\pi, \emptyset \rangle$  is coherent with  $T$ .

Again, since  $\sigma \sqsubseteq T$ ,  $\sigma$  cannot be a covering assignment (see Lemma 3.1), and hence the call **NEW-TRANSVERSAL**( $\pi$ ) cannot answer **false** at line 15. If  $\sigma$  is a witness, then **NEW-TRANSVERSAL**( $\pi$ ) returns **true** at line 16 (and the statement of the lemma would be proven).

If this is not the case, then also  $\sigma$  is a precursor of  $T$ . By Lemma E.3, there exist an edge  $G \in \text{Sep}(\sigma)$  and a free vertex  $v \in G$  of  $\sigma$  such that  $\pi' = \sigma + \langle \{v\}, G \setminus \{v\} \rangle$  is coherent with a new transversal of  $\mathcal{G}$  (again, with  $\pi \sqsubset \pi'$ ). Note that such an assignment belongs exactly to those on which a recursive call is performed at lines 17–19. Hence the statement is proven.

To conclude, let us consider the case in which  $U_\pi = \emptyset$ . In this case, **NEW-TRANSVERSAL**( $\pi$ ) does not execute the loop at lines 10–11, and the lines 12–13 do not alter in any way the sets *Included* and *Free*. This means that, after line 13, the sets *Included* and *Free* still reflect the original assignment  $\pi$  on which the procedure was called. Said so, the discussion is the very same as above, since Lemma E.3 guarantees that at least one of the recursive call performed at lines 17–19 is performed on an assignment coherent with a new transversal of  $\mathcal{G}$ .  $\square$

**Lemma E.5.** *Let  $\mathcal{G}$  and  $\mathcal{H}$  be two hypergraphs, and let  $\pi$  be an assignment. Then, the call **NEW-TRANSVERSAL**( $\pi$ ) answers **true** if and only if there exists a new transversal of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$  coherent with  $\pi$ .*

*Proof.*

( $\Rightarrow$ ) Let  $\pi = \langle \text{In}, \text{Ex} \rangle$ , and let us assume that **NEW-TRANSVERSAL**( $\pi$ ) answers **true**. Therefore, **NEW-TRANSVERSAL**( $\pi$ ) itself or one of the recursive call spawned directly or indirectly by **NEW-TRANSVERSAL**( $\pi$ ) answers **true** either at line 8 or at line 16. We have already seen that those parts of the algorithm implement exactly the checking of Condition (3) of Lemma 4.1, which we know to be a necessary and sufficient condition for an assignment to be a witness. Moreover, observe that in the algorithm the vertices of *In* are never removed from the set *Included*, and the vertices of *Ex* are never removed from the set *Excluded*. Hence, if the algorithm answers **true** it is because it has actually built a witness  $\sigma$  such that  $\pi \sqsubseteq \sigma$ . Hence, by Lemma 4.1, there exists a new transversal of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$  coherent with  $\pi$ .

( $\Leftarrow$ ) Assume now that there exists a new transversal of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$  coherent with  $\pi$ . By Lemma E.4 we know that either **NEW-TRANSVERSAL**( $\pi$ ) answers **true** at line 8 or at line 16 (and hence this direction of the

proof would be proven), or among its recursive calls there is one executed on an assignment coherent with a new transversal of  $\mathcal{G}$ .

Let us consider the latter case, and let  $\hat{\pi}^1$  be the first assignment coherent with a new transversal of  $\mathcal{G}$  on which  $\text{NEW-TRANSVERSAL}(\pi)$  executes a recursive call. Since  $\hat{\pi}^1$  is coherent with a new transversal of  $\mathcal{G}$ , Lemma E.4 applies to  $\text{NEW-TRANSVERSAL}(\hat{\pi}^1)$  too.

By the recursive application of Lemma E.4 we conclude that among all the possible sequences of recursive calls rooted in  $\text{NEW-TRANSVERSAL}(\pi)$ , there are sequences of calls successively executed on assignments coherent with a new transversal of  $\mathcal{G}$ . Let us pose, for notational convenience,  $\pi = \hat{\pi}^0$ .

Let  $\hat{p} = (\hat{\pi}^0, \hat{\pi}^1, \dots, \hat{\pi}^k)$  be the “foremost” and “longest” of those sequences such that, for all  $1 \leq i \leq k$ , the assignment  $\hat{\pi}^i$  is coherent with a new transversal of  $\mathcal{G}$  and is one of the assignments on which  $\text{NEW-TRANSVERSAL}(\hat{\pi}^{i-1})$  executes a recursive call. With “foremost” we mean that, for every  $1 \leq i \leq k$ , the assignment  $\hat{\pi}^i$  is the *first* coherent with a new transversal of  $\mathcal{G}$  on which  $\text{NEW-TRANSVERSAL}(\hat{\pi}^{i-1})$  executes a recursive call. With “longest” we mean that all the recursive calls, if any, performed by  $\text{NEW-TRANSVERSAL}(\hat{\pi}^k)$  are executed on assignments not coherent with any new transversal of  $\mathcal{G}$ . Note that such a sequence is finite because every recursive path traversed by the algorithm is finite (we have already discussed this).

Since  $\hat{\pi}^k$  is coherent with a new transversal of  $\mathcal{G}$ , and  $\hat{p}$  is the foremost and longest sequence of calls successively executed on assignments coherent with a new transversal of  $\mathcal{G}$ , from Lemma E.4 it follows that  $\text{NEW-TRANSVERSAL}(\hat{\pi}^k)$  returns **true** either at line 8 or at line 16. This positive answer propagates throughout all the recursion path and thus  $\text{NEW-TRANSVERSAL}(\pi)$  answers **true** too.  $\square$

*Proof of Theorem E.1.* By Lemma 2.3,  $\mathcal{G}$  and  $\mathcal{H}$  are dual if and only if they are simple, satisfy the intersection property, and there is no new transversal of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$ . Therefore, since  $\mathcal{G}$  and  $\mathcal{H}$  are explicitly checked to be simple hypergraphs satisfying the intersection property (line 2), the correctness of the algorithm  $\text{DET-DUAL}$  directly follows from the correctness of the procedure  $\text{NEW-TRANSVERSAL}$  (see Lemma E.5): in fact, any new transversal of  $\mathcal{G}$ , if exists, is coherent with  $\sigma_\varepsilon$  which is the assignment on which  $\text{NEW-TRANSVERSAL}$  is invoked by the procedure  $\text{DET-DUAL}$  (line 3).  $\square$

Let us now consider the time complexity of the algorithm.

**Lemma E.6.** *Let  $\mathcal{G}$  and  $\mathcal{H}$  be two hypergraphs satisfying the intersection property, and let  $\pi$  be an assignment. If  $\text{NEW-TRANSVERSAL}(\pi')$  is any of the recursive calls performed by  $\text{NEW-TRANSVERSAL}(\pi)$ , then  $|\text{Com}(\pi')| \leq \frac{1}{2}|\text{Com}(\pi)|$ .*

*Proof.* If  $\text{NEW-TRANSVERSAL}(\pi)$  does not perform any recursive call at all, then the statement of the lemma is true. Let us assume that  $\text{NEW-TRANSVERSAL}(\pi)$  performs recursive calls, and let  $U_\pi$  be the set  $U$  computed by  $\text{NEW-TRANSVERSAL}(\pi)$  at line 9. The set  $U_\pi$  can be empty or not, and we consider at first the case in which it is non-empty.

Let us focus our attention on the recursive calls performed at line 11. Simply by the definition of  $U_\pi$ , if  $v \in U_\pi$ , then  $\varepsilon_v^{\text{Com}(\pi)} \geq \frac{1}{2}$ , and thus  $|\text{Com}(\pi')| \leq \frac{1}{2}|\text{Com}(\pi)|$ , for the assignment  $\pi' = \pi + \langle \emptyset, \{v\} \rangle$  (see Lemma 3.4).

Now, let us focus our attention on the calls at line 19. If  $\text{NEW-TRANSVERSAL}(\pi)$  arrives at that stage of its execution, it means that all the vertices belonging to  $U_\pi$  have been already included. Let us denote by  $\sigma = \pi + \langle U_\pi, \emptyset \rangle$  the assignment encoded by the sets *Included* and *Excluded* at that stage of the execution flow. Observe that the set of the excluded vertices is the very same of that at the beginning of the execution of the call  $\text{NEW-TRANSVERSAL}(\pi)$ , implying that  $\text{Com}(\sigma) = \text{Com}(\pi)$ .

Since the algorithm is in the loop at lines 17–19, it is taking into consideration recursive calls on the assignments  $\pi' = \sigma + \langle \{v\}, G \setminus \{v\} \rangle$ , where  $G \in \text{Sep}(\sigma)$  and  $v \in G$  is a free vertex in  $\sigma$ . Note that  $\varepsilon_v^{\text{Com}(\sigma)} < \frac{1}{2}$  because  $v \notin U_\pi$ . Hence, by the intersection property and Lemma 3.4, it follows that  $|\text{Com}(\pi')| \leq \frac{1}{2}|\text{Com}(\pi)|$  for every assignment on which a recursive call is invoked at line 19.

To conclude, if  $U_\pi$  is empty, then only few things change. Indeed, in this case, there is no recursive call at all at line 11, and for the recursive calls at line 19 a similar discussion to the that above proves that  $|\text{Com}(\pi')| \leq \frac{1}{2}|\text{Com}(\pi)|$  (simply note that, being  $U_\pi$  empty, then  $\sigma = \pi$ ,  $v \notin U_\pi$ , and hence, again,  $\varepsilon_v^{\text{Com}(\sigma)} < \frac{1}{2}$ , and Lemma 3.4 applies).  $\square$

**Lemma E.7.** *Let  $\mathcal{G}$  and  $\mathcal{H}$  be two hypergraphs satisfying the intersection property, and let  $\pi$  be an assignment. Then, the maximum depth of every recursive path traversed by  $\text{NEW-TRANSVERSAL}(\pi)$  is  $O(\log |\text{Com}(\pi)|)$ .*

*Proof.* By Lemma E.6, it follows that the size of the set  $\text{Com}_\mathcal{H}$  halves at every recursion step, and this happens not only for those recursive calls leading to a **true** answer. Since the procedure  $\text{NEW-TRANSVERSAL}$  is correct

(Lemma E.5), every recursive path leading to a **false** answer has to return its (**false**) answer before  $\text{Com}_{\mathcal{H}}$  becomes empty, for otherwise an incorrect answer would be returned by  $\text{NEW-TRANSVERSAL}(\pi)$ . Therefore, the depth of every recursive path leading to a **false** answer is logarithmic in  $|\text{Com}(\pi)|$ .

Obviously, also the depth of a recursive path leading to a **true** answer is logarithmic in the size of  $\text{Com}(\pi)$  (again, due to the halving size of  $\text{Com}_{\mathcal{H}}$ ).  $\square$

**Theorem E.8.** *Let  $\mathcal{G}$  and  $\mathcal{H}$  be two hypergraphs satisfying the intersection property. Then, the time complexity of the algorithm DET-DUAL is  $O(N^{O(\log N)})$ .*

*Proof.* Checking whether hypergraphs  $\mathcal{G}$  and  $\mathcal{H}$  are simple and satisfy the intersection property (line 2) is feasible in  $O(N^2)$ .

Regarding the procedure NEW-TRANSVERSAL the size of the set  $U$ , computed at line 9, is bounded by  $|V|$ . Therefore there are no more than  $O(N)$  recursive calls performed at line 11. Moreover, there are no more than  $O(N^2)$  recursive calls performed at line 19, because, given any edge  $G \in \mathcal{G}$ , there are at most  $O(N)$  different vertices belonging to  $G$  and  $|\mathcal{G}|$  is  $O(N)$ .

Hence every call of the procedure NEW-TRANSVERSAL performs  $O(N^2)$  recursive calls. By Lemma E.7, every recursive path has a depth  $O(\log |\mathcal{H}|)$  that is also  $O(\log N)$ , hence at most  $O(N^{2O(\log N)})$  calls of NEW-TRANSVERSAL are executed. Since each call of NEW-TRANSVERSAL executes in time  $O(N^2)$  to perform its computations, the overall time complexity of the algorithm DET-DUAL is  $O(N^{2(O(\log N)+1)})$ , which is  $O(N^{O(\log N)})$ .  $\square$

If we assume that the input hypergraphs  $\mathcal{G}$  and  $\mathcal{H}$  are such that  $\mathcal{G} \subseteq \text{tr}(\mathcal{H})$  and  $\mathcal{H} \subseteq \text{tr}(\mathcal{G})$  (i.e., we assume the stricter condition imposed on the input hypergraphs by Boros and Makino [4]), then we can carry out a finer time complexity analysis. We need the following properties.

**Lemma E.9.** *Let  $\mathcal{H}$  be a hypergraph, and  $T = \{v_1, \dots, v_t\} \subseteq V$  be a minimal transversal of  $\mathcal{H}$ . Then, there exist (at least)  $|T|$  distinct edges  $H_1, \dots, H_t$  of  $\mathcal{H}$  such that  $T \cap H_i = \{v_i\}$ .*

*Proof.* Let  $W = \{H_1, \dots, H_p\}$  be the set of all the edges of  $\mathcal{H}$  witnessing the criticality of vertices in  $T$ . Since  $T$  is a minimal transversal of  $\mathcal{H}$ , by Lemma 2.1, for each vertex  $v \in T$  there is at least one edge in  $W$  witnessing the criticality of  $v$ . Now, assume by contradiction that  $p < t$ . Then there must exist two different vertices in  $T$ , say  $v_i$  and  $v_j$ , and an edge  $H_k \in W$  such that  $\{v_i\} = T \cap H_k$  and  $\{v_j\} = T \cap H_k$ : a contradiction, because  $v_i$  and  $v_j$  are assumed to be different.  $\square$

**Corollary E.10.** *Let  $\mathcal{H}$  be a hypergraph, and let  $T$  be a minimal transversal of  $\mathcal{H}$ . Then,  $|T| \leq |\mathcal{H}|$ .*

*Proof.* By Lemma E.9, for every minimal transversal  $T$  of size  $t$  there are at least  $t$  distinct edges of  $\mathcal{H}$  witnessing the criticality of the vertices of  $T$ . So, a minimal transversal of  $\mathcal{H}$  cannot have more vertices than the number of edges of  $\mathcal{H}$  for otherwise there would not be enough criticality's witnesses.  $\square$

**Lemma E.11.** *Let  $\mathcal{G}$  and  $\mathcal{H}$  be two hypergraphs such that  $\mathcal{G} \subseteq \text{tr}(\mathcal{H})$  or  $\mathcal{H} \subseteq \text{tr}(\mathcal{G})$ . Then,  $|V| \leq |\mathcal{G}| \cdot |\mathcal{H}|$ .*

*Proof.* If  $\mathcal{G} \subseteq \text{tr}(\mathcal{H})$ , then from Corollary E.10 it follows that, for every edge  $G \in \mathcal{G}$ ,  $|G| \leq |\mathcal{H}|$ , because  $G$  is a minimal transversal of  $\mathcal{H}$ . By summing these relations over all the edges of  $\mathcal{G}$  we obtain  $\sum_{G \in \mathcal{G}} |G| \leq |\mathcal{G}| \cdot |\mathcal{H}|$ , which, combined with  $|V| \leq \sum_{G \in \mathcal{G}} |G|$ , proves the statement. If  $\mathcal{H} \subseteq \text{tr}(\mathcal{G})$ , then the statement follows by symmetry.  $\square$

**Theorem E.12.** *Let  $\mathcal{G}$  and  $\mathcal{H}$  be two hypergraphs such that  $\mathcal{G} \subseteq \text{tr}(\mathcal{H})$  and  $\mathcal{H} \subseteq \text{tr}(\mathcal{G})$ . Then, the time complexity of the algorithm DET-DUAL, in which the condition of  $\mathcal{G}$  and  $\mathcal{H}$  being such that  $\mathcal{G} \subseteq \text{tr}(\mathcal{H})$  and  $\mathcal{H} \subseteq \text{tr}(\mathcal{G})$  is checked, instead of them being simple and satisfying the intersection property, is  $O((|\mathcal{G}| \cdot |\mathcal{H}|)^{O(\log |\mathcal{H}|)})$ .*

*Proof.* To prove the statement we need a similar proof to that of Theorem E.8. Just observe that verifying that  $\mathcal{G}$  and  $\mathcal{H}$  are such that  $\mathcal{G} \subseteq \text{tr}(\mathcal{H})$  and  $\mathcal{H} \subseteq \text{tr}(\mathcal{G})$  is feasible in  $O(N^2)$ . Moreover, by Lemma E.11, the size of the set  $U$ , computed at line 9, is bounded by  $|\mathcal{G}| \cdot |\mathcal{H}|$ . In addition, there are no more than  $|\mathcal{G}| \cdot |\mathcal{H}|$  recursive calls performed at line 19, because given any edge  $G \in \mathcal{G}$ , from  $\mathcal{G} \subseteq \text{tr}(\mathcal{H})$  and Corollary E.10 it follows that there are at most  $|\mathcal{H}|$  different vertices belonging to  $G$ .

Therefore, every call of the procedure NEW-TRANSVERSAL performs  $O(|\mathcal{G}| \cdot |\mathcal{H}|)$  recursive calls. Since each call executes actually in time  $O(|\mathcal{G}| \cdot |\mathcal{H}|)$  to perform its computations, the overall time complexity of the algorithm DET-DUAL (with the modifications mentioned in the statement of the lemma) is  $O((|\mathcal{G}| \cdot |\mathcal{H}|)^{O(\log |\mathcal{H}|)+1)})$ , which is  $O((|\mathcal{G}| \cdot |\mathcal{H}|)^{O(\log |\mathcal{H}|)})$ .  $\square$

It is now interesting to focus our attention on the following fact. By the proof of Theorem E.8, the leaves of the recursion tree of the procedure NEW-TRANSVERSAL, which are candidate to certify the existence of a new transversal of  $\mathcal{G}$ , are  $O(N^{2O(\log N)})$ , but in principle there could be an exponential number of new minimal transversals of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$ . For example, let us consider the class of pairs of hypergraphs  $\{(\mathcal{G}_i, \mathcal{H}_i)\}_{i \geq 1}$ , defined

as follows:  $V_i = \{x_1, y_1, \dots, x_i, y_i\}$ ,  $\mathcal{G}_i = \{\{x_j, y_j\} \mid 1 \leq j \leq i\}$ , and  $\mathcal{H}_i = \{\{x_1, \dots, x_i\}, \{y_1, \dots, y_i\}\}$ . For every  $i \geq 1$ , the hypergraphs  $\mathcal{G}_i$  and  $\mathcal{H}_i$  satisfy the intersection property,  $|\mathcal{G}_i| = i$ ,  $|\mathcal{H}_i| = 2$ , and the number of minimal transversals of  $\mathcal{G}_i$  missing in  $\mathcal{H}_i$  is  $\Theta(2^i)$ . So, it is not possible that each leaf of the recursion tree of NEW-TRANSVERSAL identifies a unique new minimal transversal of  $\mathcal{G}$ . For this reason we want to know what new transversals of  $\mathcal{G}$  are identified by NEW-TRANSVERSAL when it answers **true**.

**Theorem E.13.** *Let  $\mathcal{G}$  and  $\mathcal{H}$  be two hypergraphs. If NEW-TRANSVERSAL( $\sigma_\varepsilon$ ) answers **true**, then the witness  $\sigma$  on which the procedure has answered **true** at the end of its recursion is coherent with a new minimal transversal of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$ .*

*Proof.* Since  $\sigma$  is an assignment passing the test of Condition (3) of Lemma 4.1, by the very same lemma  $\sigma$  must be coherent with a new transversal of  $\mathcal{G}$ , say,  $\tilde{T}$ . Let us assume by contradiction that  $\sigma$  is not coherent with any minimal one, and let  $\hat{T}$  be any new minimal transversal *strictly* contained in  $\tilde{T}$ .

Let  $(\pi^0, \pi^1, \dots, \pi^k)$  be the sequence of the assignments successively considered by the stack of the recursive calls of NEW-TRANSVERSAL resulting in the construction of the witness  $\sigma$ , where  $\pi^0 = \sigma_\varepsilon = \langle \emptyset, \emptyset \rangle$ ,  $\pi^k = \sigma$ , and  $\pi^i = \langle In^i, Ex^i \rangle$  for all  $0 \leq i \leq k$ .

Remember that assignments  $\pi^i$  are such that  $\pi^\ell \sqsubset \pi^{\ell+1}$ , for every  $0 \leq \ell \leq k-1$  (see Lemma E.4). This implies that, for every  $0 \leq i \leq k$ , assignment  $\pi^i$  is such that  $\pi^i \sqsubseteq \pi^k = \sigma \sqsubseteq \tilde{T}$ , and hence  $Ex^i \cap \hat{T} = \emptyset$  (because  $Ex^i \cap \tilde{T} = \emptyset$  and  $\hat{T} \subset \tilde{T}$ ). For this reason, in order for  $\pi^k = \sigma$  not to be coherent with  $\hat{T}$  it must be the case that  $In^k \not\subseteq \hat{T}$ . Since  $\emptyset = In^0 \subseteq \hat{T}$ ,  $In^k \not\subseteq \hat{T}$ , and  $In^\ell \subseteq In^{\ell+1}$ , for every  $0 \leq \ell \leq k-1$ , there must exist an index  $j$ , with  $0 \leq j \leq k-1$ , such that  $In^j \subseteq \hat{T}$  and  $In^{j+1} \not\subseteq \hat{T}$ .

Let  $v$  be a vertex belonging to  $In^{j+1} \setminus \hat{T}$ . This means that  $v \notin \hat{T}$ , and from  $In^j \subseteq \hat{T}$  it follows that  $v \notin In^j$ . Therefore, the vertex  $v$  is included during the call NEW-TRANSVERSAL( $\pi^j$ ) either at lines 12–13, or during a recursive call performed at line 19.

Consider at first the latter case. If  $v$  were included at line 19, then it would be included as a critical vertex along with an edge of  $\mathcal{G}$ , say  $G$ , chosen at that moment, witnessing the criticality of  $v$  in  $In^{j+1}$ . This means that  $v \in In^{j+1}$ , and  $(G \setminus \{v\}) \subseteq Ex^{j+1}$ . From  $\pi^{j+1} \sqsubseteq \pi^k \sqsubseteq \tilde{T}$  it follows that  $v \in \tilde{T}$  and that  $\tilde{T} \cap (G \setminus \{v\}) = \emptyset$  (because  $\tilde{T} \cap Ex^{j+1} = \emptyset$ ), and hence that  $\tilde{T} \cap G = \{v\}$ . Therefore  $v$  is critical in  $\tilde{T}$ . But,  $v \notin \hat{T}$  and we are assuming that  $\hat{T}$  is a minimal transversal of  $\mathcal{G}$  such that  $\hat{T} \subset \tilde{T}$ : a contradiction, because  $v$  is at the same time critical and non-critical in  $\tilde{T}$ . As a consequence,  $v$  has to be included at lines 12–13, as part of the set  $U_{\pi^j}$  during the call NEW-TRANSVERSAL( $\pi^j$ ).

Since the execution flow of NEW-TRANSVERSAL( $\pi^j$ ) goes beyond line 11, all the recursive calls performed at line 11 return **false**. Among those calls, also NEW-TRANSVERSAL( $\pi^j + \langle \emptyset, \{v\} \rangle$ ) is performed (because  $v \in U_{\pi^j}$ ), and the fact that it returns **false** implies, by Lemma E.5, that there is no new transversal of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$  coherent with  $\pi^j + \langle \emptyset, \{v\} \rangle$ . But, note that  $In^j \subseteq \hat{T}$ , and that  $(Ex^j \cup \{v\}) \cap \hat{T} = \emptyset$  (because  $Ex^j \cap \hat{T} = \emptyset$  and  $v \notin \hat{T}$ ). Hence,  $\hat{T}$  is a new transversal of  $\mathcal{G}$  coherent with  $\pi^j + \langle \emptyset, \{v\} \rangle$ : a contradiction, because we are assuming that NEW-TRANSVERSAL( $\pi^j + \langle \emptyset, \{v\} \rangle$ ) returns **false**. Thus  $\sigma$  is coherent with a new minimal transversal of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$ .  $\square$

Let  $\mathcal{G}$  and  $\mathcal{H}$  be two hypergraphs. Assume that NEW-TRANSVERSAL( $\sigma_\varepsilon$ ) answers **true**, and let  $\sigma = \langle In, Ex \rangle$  be the assignment on which the procedure answers **true** at the end of its recursion. There are two cases: (a)  $Sep(\sigma) = \emptyset$ , or (b)  $Com(\sigma) = \emptyset$ .

In Case (a), by Theorem E.13,  $\sigma$  is coherent with a new minimal transversal of  $\mathcal{G}$ , and it is such that  $Sep(\sigma) = \emptyset$ , hence it follows that  $In$  is a new minimal transversal of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$ .

Let us consider Case (b). We claim that the set  $T_\sigma = \{In \cup S \mid S \in tr(Sep(\sigma)^{Free(\sigma)})\}$  is a set of new minimal transversals of  $\mathcal{G}$  w.r.t.  $\mathcal{H}$ . This easily follows from an adaptation of Lemma D.4 to  $Sep(\sigma)$ . Note that all the elements of  $T_\sigma$  are new minimal transversals of  $\mathcal{G}$ , but there is no guarantee that there are not more, i.e.,  $(\mathcal{H} \cup T_\sigma) \subseteq tr(\mathcal{G})$  and it could even be the case that  $(\mathcal{H} \cup T_\sigma) \subset tr(\mathcal{G})$ .